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Improved Operational Matrices of DP-Ball Polynomials for Solving Singular Second Order Linear Dirichlet-type Boundary Value Problems

Ahmed Kherd[1*], Salim F. Bamsaoud[2], Omer Bazighifan[3, 4], Mobarek A. Assabaai[3]

[1]Department of Mathematics, Faculty of Computer Science & Engineering, Al-Ahgaff University, Mukalla, Yemen.
[2]Department of Physics, College of Sciences, Hadhramout University, Mukalla, Yemen.
[3]Department of Mathematics, College of Sciences, Hadhramout University, Mukalla, Yemen.
[4]Department of Mathematics, College of Education, Seiyun University, Hadhramout, Yemen.

*Corresponding authors: khrd@ahgaff.edu

Abstract: Solving Dirichlet-type boundary value problems (BVPs) using a novel numerical approach is presented in this study. The operational matrices of DP-Ball Polynomials are used to solve the linear second-order BVPs. The modification of the operational matrix eliminates the BVP's singularity. Consequently, guaranteeing a solution is reached. In this article, three different examples were taken into consideration in order to demonstrate the applicability of the method. Based on the findings, it seems that the methodology may be used effectively to provide accurate solutions.

Keywords: Boundary value problems; Dirichlet-type; linear second-order; Operational matrices; Singular boundary value problems.

1. Introduction

A numerical approach for addressing singular second-order linear boundary value problems of the Dirichlet type is provided. These kinds of problems occur in different applications, such as structural mechanics, chemical reactions, and gas dynamics. The existence and uniqueness of the solution for such problems were described in [1]. Series solutions, Chebyshev polynomials, B-splines, and cubic splines [2]–[5] have been taken into consideration by a number of investigators to solve these kinds of BVPs. Moreover, additional methods include fitted mesh [6], Green's functions and decomposition [7] and Green's matrix [8]. Methods which are based on reproducing kernel space [9], [10], Sinc collocation method [11], Sinc Galerkin method [12], and an iterative predictor-corrector type method which is based on finite difference approximation [13] are also included in latest results. There have been some reviews of existing methods based on Bernstein functions published in [14] and [15]. Chebyshev collocation method [16] was presented to solve the singular two-point boundary value problems of differential equations. The Adomian decomposition method is used to solve a class of singular differential equations with Dirichlet conditions [17]. Dejdumrong operational matrix [18] was applied to obtain the solutions of some types of differential equations. Haar wavelet collocation method was introduced to get the solution of Lane-Emden equations with Dirichlet and another type of boundary conditions [19]. Lately, there are some authors who have studied the given problem with Dirichlet boundary conditions [20, 21, 22].

This article aims to develop an improved operational matrix as a numerical method for solving singular second-order Dirichlet-type boundary value problems. Asides from this introductory section, a review of the Ball polynomial is presented in Section 2, with applications of the operational matrix of derivative explained in Section 3. Relevant numerical problems are considered in Section 4, where the results and comparison with other authors are shown. The article is concluded in Section 5.

2. Review on Ball Polynomial

The Ball polynomial was declared by A. A. Ball in his well-known aircraft design system CONSURF in [23]**.** It is described as a cubic polynomial and explained mathematically as

$$(1-t)^2, 2t(1-t)^2, 2t^2(1-t), t^2, \quad 0 \le t \le 1. \qquad (1)$$

The high generality of the Ball polynomial has been the subject of discussion in a number of recent papers, as well as its properties. For example, in the 1980s, there were two different Ball polynomials of arbitrary degree. These polynomials, which were given the names Said-Ball and Wang-Ball, [24]-[26] and DP-Ball, were another generalization of the Ball polynomial that came out in 2003 [27].

A. DP-Ball Polynomial Representation

The degree m DP-Ball polynomial [27] is defined by:

$$\mathcal{D}_i^m(t) = \begin{cases} (1-t)^m & , i = 0, \\ t(1-t)^{m-i} & , 1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor - 1, \\ \mathcal{K}_1^m(t) + \mathcal{K}_2^m(t) & , i = \left\lfloor \frac{m}{2} \right\rfloor, \\ \mathcal{K}_1^m(t) + \mathcal{K}_3^m(t) & , i = \left\lceil \frac{m}{2} \right\rceil, \\ \mathcal{D}_{m-i}^m(1-t) & , \left\lceil \frac{m}{2} \right\rceil + 1 \le i \le m, \end{cases} \qquad (2)$$

and

$$\mathcal{K}_1^m(t) = \left(\frac{1}{2}\right)^{\left\lceil \frac{1}{2} \right\rceil - \left\lfloor \frac{1}{2} \right\rfloor} \left(1 - t^{\left\lfloor \frac{1}{2} \right\rfloor + 1} - (1-t)^{\left\lfloor \frac{1}{2} \right\rfloor + 1}\right),$$

$$\mathcal{K}_2^m(t) = \left(\left\lceil \frac{1}{2} \right\rceil - \left\lfloor \frac{1}{2} \right\rfloor\right) t(1-t)^{\left\lfloor \frac{1}{2} \right\rfloor + 1},$$

$$\mathcal{K}_3^m(t) = \left(\left\lceil \frac{1}{2} \right\rceil - \left\lfloor \frac{1}{2} \right\rfloor\right) t^{\left\lfloor \frac{1}{2} \right\rfloor + 1}(1-t).$$

$\lfloor t \rfloor$ and $\lceil t \rceil$ denote the greatest integer less than or equal to, and the least integer greater than or equal to t , respectively.

Definition:
The DP Monomial matrix form can be formulated in the form by [28]

$$\mathcal{D} = \begin{bmatrix} d_{00} & d_{01} & \cdots & \cdots & d_{0m} \\ d_{10} & d_{11} & \cdots & \cdots & d_{1m} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ d_{m0} & d_{m1} & \cdots & \cdots & d_{mm} \end{bmatrix}_{(m+1)\times(m+1)} , \qquad (3)$$

where

$$d_{ij} = \begin{cases} (-1)^j \binom{n}{j}, \text{for } i = 0, \\ (-1)^{j-1} \binom{n-i}{j-1}, \text{for } 0 < i £ \left\lfloor \frac{n}{2} \right\rfloor - 1, \\ (-1)^{j-1}(n-2i)\binom{i+1}{j-1} + \frac{1}{2}^{n-2i} \left(\binom{0}{j} \right. \\ \left. -\binom{0}{j-i-1} - (-1)^j \binom{i+1}{j}\right), \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor, \\ (-1)^{j-n+i}(n-2i)\binom{1}{j-n+i-1} + \frac{1}{2}^{2i-n} \left(\binom{0}{j} \right. \\ \left. -\binom{0}{j-n+i-1} - (-1)^j \binom{n-i+1}{j}\right), \text{for } i = \left\lceil \frac{n}{2} \right\rceil, \\ (-1)^{(j-i)} \binom{1}{j-i} 1, \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n-1, \\ \binom{0}{j-n}, \text{for } i = n. \end{cases} \qquad (4)$$

The following are some of the properties that are satisfied by the DP-Ball basis function:

i. The DP-Ball basis function is non-negative; that is,

$$\mathcal{D}_i^m(t) \ge 0, \forall i = 0,1,\cdots,m. \qquad (5)$$

ii. The partition of unity that is,

$$\sum_{i=0}^m \mathcal{D}_i^m(t) = 1. \qquad (6)$$

In general, we approximate any function $y(t)$ with the first $(m+1)$ DP-Ball polynomials as:

$$y(t) \approx \sum_{i=0}^m c_i \mathcal{D}_i^m(t) = C^T\phi(t) = C^T\mathcal{D}T(t), \qquad (7)$$

where $C^T = [c_0, c_1, \dots, c_m], T(t) = [1 \quad t \quad \dots t^m]^T$ and $\mathcal{D}$ is the DP monomial matrix form given in (3). The operational matrix of the derivative of the DP-Ball polynomials set $\phi(t)$ is given by

$\frac{d\phi(t)}{dx} = D^{(1)}\phi(t)$ is the $m+1$ by $m+1$ operational matrix of the derivative is defined as:

$$\begin{aligned} \frac{d}{dx}\phi(t) &= \frac{d}{dx}(X) \\ &= \mathcal{D}\frac{d}{dx}X \\ &= (\mathcal{D}V\mathcal{D}^{-1})\mathcal{D}X \\ &= D^{(1)}\phi(x). \end{aligned} \qquad (8)$$

Hence

$$D^{(1)} = \mathcal{D}V\mathcal{D}^{-1}, \qquad (9)$$

where D is DP-Ball monomial matrix form, and

$$V = \begin{cases} j & , i = j+1. \\ 0 & , \text{otherwise}. \end{cases} \qquad (10)$$

We can generalize Equation (9) as

$$\frac{d^n}{dt^n}\phi(t) = \frac{d^{n-1}}{dt^{n-1}}\left(\frac{d}{dt}\phi(t)\right) = \frac{d^{n-1}}{dt^{n-1}}\left(D^{(1)}\phi(t)\right) = \cdots = (D^{(1)})^n\phi(t) = D^{(n)}\phi(t), \qquad n = 1,2,\dots$$

B. Practical Implementation of the Derivational Operational Matrix:

Within this part, we will provide the derivation of the technique for solving differential equations of the type

$$p_0(t)y''(t) + p_1(t)y'(t) + p_2(t)(y(t))^n = g(t), \quad (11)$$

with Dirichlet boundary conditions

$$y(0) = \alpha_1, \quad y(1) = \alpha_2. \qquad (12)$$

Where $p_j(t)$, $j = 0,1,2$, $\alpha_i, i = 1,2$ and $g(t)$ are known, while $y(t)$ is unknown.
Approximating Equation (11) by DP-Ball Polynomials as follows:

$$p_0(t)C^TD^{(2)}\phi(t) + p_1(t)C^TD^{(1)}\phi(t) + p_2(t)\left(C^T\phi(t)\right)^n = G^T\phi(t). \qquad (13)$$

Where $G^T = [g_0, g_1, \cdots, g_m]$, we can write the residual $\mathfrak{R}(t)$ for Equation (13) as

$$\mathfrak{R}(t) = p_0(t)C^TD^{(2)}\phi(t) + p_1(t)C^TD^{(1)}\phi(t) + p_2(t)(C^T\phi(t))^n - G^T\phi(t). \qquad (14)$$

To find the solution of $y(t)$ given in (11), we first collocate (14) at $(m-1)$ points. For suitable collection points, we use $t_i = \frac{1}{2} - \frac{1}{2}\left[\cos\left(\frac{(i+1)\pi}{m+1}\right)\right], i = 0,1,\cdots,m-1.$ Together, these equations with (12) generate (m+1) nonlinear equations that can be solved using Newton's iteration approach. As a consequence of this, $y(t)$ may be calculated.

3. Numerical Problems:

Problem 1:

$$y''(t) + \frac{1}{t}y'(t) + y(t) = 4 - 9t + t^2 - t^3,$$
$$y(0) = 0, \quad y(1) = 0. \qquad (15)$$

Exact Solution:

$$y(t) = t^2 - t^3.$$

Source: [29].

Equation (15) may be solved by using our method with the parameter $(m = 3)$, we get $c_0 = 0$, $c_1 = \frac{-1}{3}$, $c_2 = \frac{2}{3}$ and $c_3 = 0$. The approximate solution as

$$y(t) \approx y_3(t)$$
$$= C^T\phi(t)$$
$$= [c_0, c_1, c_2, c_3][\mathcal{D}_0^3(t), \mathcal{D}_1^3(t), \mathcal{D}_2^3(t), \mathcal{D}_3^3(t)]^T$$
$$= c_0\mathcal{D}_0^3(t) + c_1\mathcal{D}_1^3(t) + c_2\mathcal{D}_2^3(t) + c_3\mathcal{D}_3^3(t)$$
$$= \begin{bmatrix} 0 & -\frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix} \begin{bmatrix} -(t-1)^3 \\ t(t-1)(t-2) \\ t(1-t^2) \\ t^3 \end{bmatrix}$$
$$= t^2 - t^3 \qquad (16)$$

Problem 2:

The following form of a singular boundary value problem of the Dirichlet type on the interval [0, 1] is taken into consideration

$$y''(x) - \frac{1}{x}y' + \frac{1}{x(1+x)}y(x) = -x^3,$$
$$y(0) = 0, y(1) = 0. \qquad (17)$$

Exact Solution:

$$y(x) = \frac{1}{144(-1+2\ln(2))}(14\ln(x+1)x + 14\ln(x+1) - 14x + 6x^2 - 12x^2\ln(2) - 2x^3 + 4x^3\ln(2) + x^4 - 2x^4\ln(2) + 9x^5 - 18x^5\ln(2))$$

Source: [30].

We apply the proposed method above to get the solutions when $m = 9$ as follows:

$$y_9(x) = 0.000000004x + 0.084172400x^2 - 0.028045963x^3 + 0.013921562x^4 - 0.0745726588x^5 + 0.0069360632x^6 - 0.0033157327x^7 + 0.0010688217x^8 - 0.00016449544x^9.$$

Figures (I, II) indicate the absolute error and numerical solutions, respectively, for problem 2.

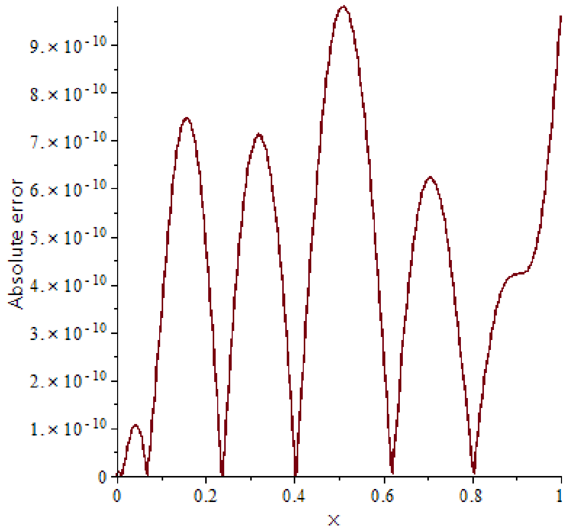

Fig. I. Absolute error with $m = 9$ for problem 2

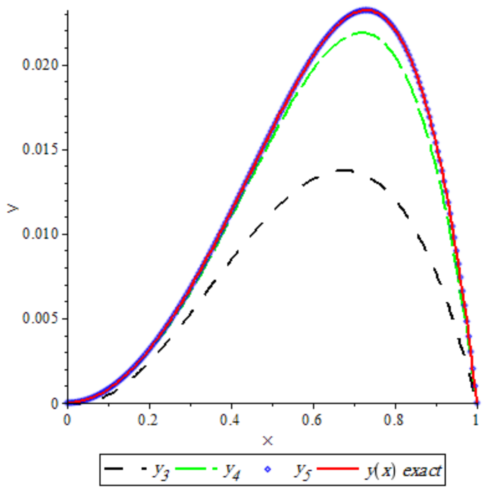

Fig. II. Numerical solutions $y_m, m = 3, 4, 5$ and exact solution y_e for problem 2

Problem 3:

$$y''(x) + \frac{2}{x}y'(x) + y(x) = 0,$$
$$y(0) = 1, \quad y(1) = \sin(1). \qquad (15)$$

Exact Solution:

$$y(x) = \frac{\sin(x)}{x}.$$

Source: [17].

Applying the proposed method for $m = 10$, the table I illustrates the numerical results of the presented method (PM) in comparison with the exact solution and other methods [17].

Table (1) shows that the absolute error was largest when $x_i = 2.0$ for both the Standard Adomian Decomposition Method (SADM) and the presented method. The SADM approach produced zero values to the fourth decimal point, but the given method gave zeros to the eighth decimal point.

Table 1. Comparison the absolute error for problem 3 with m = 10.

x_i	Exact solution	Absolute Error MADM	Absolute Error SADM	Absolute Error PM
1.0	0.841470984807897	0.0E-7	3.00E-8	1.77685E-18
1.1	0.810188509146759	0.0E-7	6.00E-8	2.89671E-13
1.2	0.776699238306022	0.0E-7	1.50E-7	3.43581E-12
1.3	0.741198604167072	0.0E-7	3.50E-7	2.32881E-11
1.4	0.703892664277472	0.0E-7	7.20E-7	1.14019E-10
1.5	0.664996657736036	0.0E-7	1.42E-6	4.47034E-10
1.6	0.624733501900941	0.0E-7	2.71E-6	1.48829E-09
1.7	0.583332241442629	0.0E-7	4.96E-6	4.36725E-09
1.8	0.541026461598997	0.0E-7	8.76E-6	1.15860E-08
1.9	0.498052677730218	0.0E-7	1.50E-5	2.82985E-08
2.0	0.454648713412841	0.0E-7	2.50E-5	6.45020E-08

All absolute error values for the Modified Adomian Decomposition method (MADM) were published only to the seventh decimal point (see the third column in table I). This clearly demonstrates that the proposed method produced excellent outcomes.

4. Conclusion:

In the current article, a new numerical method is given to determine the solution for linear problems with a single boundary value. The utilization of DP-Ball Polynomials allows for an approximation of the unknown function while also treating boundary conditions of the Dirichlet type. In addition to that, it may also be used in situations involving singular boundary value issues. In this article, three numerical examples are used to show how useful the method is.

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المصفوفات التشغيلية المحسنة لمتعدد حدود دي – بي بول لحل مسائل القيمة الحدودية من نوع ديريتشليت الشاذة من الدرجة الثانية الخطية

أحمد خرد[1*]، سالم فرج بامسعود[2]، عمر بازغيفان[3،4]، مبارك عوض السباعي[3]

الخلاصة: تم حل مشكلات قيمة الحدود من نوع ديريتشليت باستخدام نهج رقمي جديد في هذه الدراسة. تُستخدم المصفوفات التشغيلية لعناصر الحدود التي تعمل في متعدد حدود دي – بي بول لحل BVPs من الدرجة الثانية. تعديل المصفوفة التشغيلية يلغي الشذوذ .BVP وبالتالي، يتم الوصول إلى حل. في هذه المقالة، تم أخذ ثلاث مسائل مختلفة في الاعتبار من أجل إثبات قابلية تطبيق الطريقة. بناءً على النتائج، يبدو أنه يمكن استخدام المنهجية بفعالية لتوفير حلول دقيقة.

الكلمات المفتاحية: مسائل القيمة الحدية؛ نوع ديريتشليت، الدرجة الثانية الخطية، المصفوفات التشغيلية؛ مسائل قيمة الحدود الشاذة