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Fourth, Fifth and Seventh-Order Iterative Methods for Solving Nonlinear Equations

Hassan M. S. Bawazir

Department of Mathematics, Faculty of Education, Seiyun University.

Corresponding author: hbawazir@yahoo.com

Abstract: In this paper, three iterative techniques are introduced for finding the numerical solutions of nonlinear equations. It is demonstrated that our methods exhibit convergence of four, five, and seven orders. Analysis reveals that our approaches outperform certain recent methods in terms of efficiency. The performance of the proposed techniques is evaluated through numerical examples. Additionally, the theoretical order of convergence is confirmed through these examples.

Key Words: Nonlinear equation, iterative method, Newton's method, convergence order.

1. Introduction:

Nonlinear equation root finding plays a significant role in numerical analysis and has wide-ranging applications in science and engineering [1, 2]. High precision is often necessary for numerical computations, highlighting the importance of higher-order numerical methods [3].

The development of a method to tackle the non-linear equation $f(x)=0$ is imperative in numerical analysis, given the frequent occurrence of such equations in diverse fields such as science, technology, and engineering [4].

The solution of various equations, such as differential and integral equations, typically involves tackling nonlinear equations [5, 6]. Within this article, we propose the introduction of three innovative iterative techniques for locating a simple root λ of a nonlinear equation $f(x)=0$, where $f: I \subset R \to R$: is a scalar function on an open interval I .

The Newton's method (NM):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0,1,2,... \quad , \tag{1}$$

well-known and extensively utilized, provides a sequential series of approximations that converge quadratically to a simple root λ of the equation $f(x)=0$.

Traub, [7], commenced the classification of iterative methodologies, advocating for a third-order iterative process. Jarrat, [8], proposed a series of methods including two points and two steps, with one function and two derivative assessments per iteration, and one parameter for achieving fourth-order convergence.

In recent years, a multitude of authors have formulated high-order iterative methods and scrutinized their convergence analysis when applied to the solution of nonlinear equations, as evidenced by [10-14] and the associated references. We present and conduct analysis of new three steps, fourth- order and fifth-order iterative methods and a four steps, seventh-order iterative method for the solution of nonlinear equations.

The convergence order of a sequence $<x_n>$ to a simple zero, λ , of a real function $f(x)$ is a positive real number α if the limit $\lim_{n\to\infty}\left(|x_{n+1}-\lambda|/|x_n-\lambda|^{\alpha}\right)=\beta$ is satisfied, where $\beta \in R^{+}$, [15]. If $\alpha = 2$ or 3 the sequence is said to have quadratic convergence or cubic convergence, respectively.

We call the relation

$$\varepsilon_{n+1} = \beta\varepsilon_n^{\alpha} + O(\varepsilon_n^{\alpha+1}) \qquad (2)$$

as the error equation, where $\varepsilon_n = x_n - \lambda$ is the error in the n-th iteration. If we can demonstrate the error equation for any iterative method, then the value of α is its order of convergence, [16].

Consider three successive iterations x_{n+1}, x_n and x_{n-1}, that are closer to the root λ, then, the computational order of convergence ρ, see [17], is approximated by using (2) as

$$\rho = \frac{\ln|(x_{n+1}-\lambda)/(x_n-\lambda)|}{\ln|(x_n-\lambda)/(x_{n-1}-\lambda)|}.$$

The efficiency index can be calculated as $p^{1/w}$, where p is the order of the method and w is the number of function evaluations per iteration required by the method, [18].

At the start, we present three newly developed higher-order methods and establish the convergence orders associated with these techniques. In conclusion, various numerical examples are provided to validate the theoretical results and illustrate their performance.

2. Main Results:

Firstly, we introduce new fourth-order iterative method (4thOIM) as follows

$$\begin{aligned} y_n &= x_n - f(x_n)/f'(x_n),\ \tilde{y}_n = x_n + f(x_n)/f'(x_n),\\ x_{n+1} &= y_n + \frac{2f(x_n)f(y_n)}{f'(x_n)(5f(y_n)-f(\tilde{y}_n))},\quad n=0,1,2,\ldots \end{aligned} \qquad (3)$$

For the method (3) we have the following convergence result.

Theorem 1 Let λ be a simple zero of sufficiently differentiable function $f: I \subseteq R \to R$ for an open interval I. If x_0 is sufficiently close to λ, then the method defined by (3) is of fourth-order and satisfies the error equation

$$\varepsilon_{n+1} = c_2^3\varepsilon_n^4 + O(\varepsilon_n^5)$$

where $\varepsilon_n = x_n - \lambda,\ c_k = f^{(k)}(\lambda)/(k!\,f'(\lambda))$.

Proof.

Using Taylor expansion of $f(x_n)$ and $f'(x_n)$ about λ, we get

$$f(x_n) = f'(\lambda)[\varepsilon_n + c_2\varepsilon_n^2 + c_3\varepsilon_n^3 + O(\varepsilon_n^4)], \qquad (4)$$

and

$$f'(x_n) = f'(\lambda)[1 + 2c_2\varepsilon_n + 3c_3\varepsilon_n^2 + O(\varepsilon_n^3)], \qquad (5)$$

therefore

$$\begin{aligned} &f(x_n)/f'(x_n)\\ &= \varepsilon_n - c_2\varepsilon_n^2 + 2(c_2^2 - c_3)\varepsilon_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)\varepsilon_n^4\\ &\quad + (6c_3^2 + 10c_2c_4 + 8c_2^4 - 4c_5 - 20c_2^2c_3)\varepsilon_n^5 + (17c_3c_4\\ &\quad + 13c_2c_5 + 52c_2^3c_3 - 28c_2^2c_4 - 5c_6 - 33c_2c_3^2 - 16c_2^5)\varepsilon_n^6\\ &\quad + (16c_2c_6 - 6c_7 - 36c_2^2c_5 + 22c_3c_5 - 92c_2c_3c_4 + 70c_2^3c_4\\ &\quad + 12c_4^2 + 126c_2^2c_3^2 - 18c_3^3 - 128c_2^4c_3 + 32c_2^6)\varepsilon_n^7 + O(\varepsilon_n^8) \end{aligned}, \qquad (6)$$

so

$$\begin{aligned} d_n &= y_n - \lambda = \varepsilon_n - f(x_n)/f'(x_n)\\ &= c_2\varepsilon_n^2 + 2(c_3 - c_2^2)\varepsilon_n^3 + (4c_2^3 + 3c_4 - 7c_2c_3)\varepsilon_n^4 + (4c_5 + 20c_2^2c_3\\ &\quad - 6c_3^2 - 10c_2c_4 - 8c_2^4)\varepsilon_n^5 + (28c_2^2c_4 + 5c_6 + 33c_2c_3^2 + 16c_2^5\\ &\quad - 17c_3c_4 - 13c_2c_5 - 52c_2^3c_3)\varepsilon_n^6 + (6c_7 - 16c_2c_6 + 36c_2^2c_5\\ &\quad - 22c_3c_5 + 92c_2c_3c_4 - 70c_2^3c_4 - 12c_4^2 - 126c_2^2c_3^2 + 18c_3^3\\ &\quad + 128c_2^4c_3 - 32c_2^6)\varepsilon_n^7 + O(\varepsilon_n^8) \end{aligned}, \qquad (7)$$

and

$$\begin{aligned}\tilde{d}_n &= \tilde{y}_n - \lambda = \varepsilon_n + f(x_n)/f'(x_n)\\ &= 2\varepsilon_2 - d_n\\ &= 2\varepsilon_2 - c_2\varepsilon_n^2 + 2(c_2^2 - c_3)\varepsilon_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)\varepsilon_n^4 + (6c_3^2 + 10c_2c_4\\ &\quad + 8c_2^4 - 4c_5 - 20c_2^2c_3)\varepsilon_n^5 + (17c_3c_4 + 13c_2c_5 + 52c_2^3c_3\\ &\quad - 28c_2^2c_4 - 5c_6 - 33c_2c_3^2 - 16c_2^5)\varepsilon_n^6 + (16c_2c_6 - 6c_7 - 36c_2^2c_5\\ &\quad + 22c_3c_5 - 92c_2c_3c_4 + 70c_2^3c_4 + 12c_4^2 + 126c_2^2c_3^2 - 18c_3^3\\ &\quad - 128c_2^4c_3 + 32c_2^6)\varepsilon_n^7 + O(\varepsilon_n^8)\end{aligned} \qquad (8)$$

Taylor expansions of $f(y_n)$ and $f(\tilde{y}_n)$ around λ are given as

$$f(y_n) = f'(\lambda)[d_n + c_2d_n^2 + c_3d_n^3 + O(d_n^4)]\ , \qquad (9)$$

$$f(\tilde{y}_n) = f'(\lambda)[\tilde{d}_n + c_2\tilde{d}_n^2 + c_3\tilde{d}_n^3 + O(\tilde{d}_n^4)]\ , \qquad (10)$$

and hence, by (7) and (8), we attain

$$\begin{aligned}&f(y_n)\\ &= f'(\lambda)[c_2\varepsilon_n^2 + 2(c_3 - c_2^2)\varepsilon_n^3 + (5c_2^3 + 3c_4 - 7c_2c_3)\varepsilon_n^4\\ &\quad + (24c_2^2c_3 + 4c_5 - 12c_2^4 - 6c_3^2 - 10c_2c_4)\varepsilon_n^5 + (28c_2^5\\ &\quad + 34c_2^2c_4 + 5c_6 + 37c_2c_3^2 - 73c_2^3c_3 - 17c_3c_4 - 13c_2c_5)\varepsilon_n^6\\ &\quad + (6c_7 - 16c_2c_6 + 44c_2^2c_5 - 22c_3c_5 + 104c_2c_3c_4 - 102c_2^3c_4\\ &\quad - 12c_4^2 - 160c_2^2c_3^2 + 18c_3^3 + 206c_2^4c_3 - 64c_2^6)\varepsilon_n^7 + O(\varepsilon_n^8)]\end{aligned}\ , \qquad (11)$$

and

$$\begin{aligned}&f(\tilde{y}_n)\\ &= f'(\lambda)[2\varepsilon_n + 3c_2\varepsilon_n^2 + 2(3c_3 - c_2^2)\varepsilon_n^3 + (5c_2^3 + 13c_4 - 13c_2c_3)\varepsilon_n^4\\ &\quad + (42c_2^2c_3 + 28c_5 - 12c_2^4 - 18c_3^2 - 34c_2c_4)\varepsilon_n^5 + (28c_2^5\\ &\quad + 106c_2^2c_4 + 59c_6 + 103c_2c_3^2 - 119c_2^3c_3 - 83c_3c_4 - 83c_2c_5)\varepsilon_n^6 + O(\varepsilon_n^7)\end{aligned}\ , \qquad (12)$$

Now, from (4), (11) and (12), we have

$$\begin{aligned}&f(x_n)f(y_n)\\ &= f'^2(\lambda)[c_2\varepsilon_n^3 + (2c_3 - c_2^2)\varepsilon_n^4 + (3c_2^3 + 3c_4 - 4c_2c_3)\varepsilon_n^5\\ &\quad + (15c_2^2c_3 + 4c_5 - 7c_2^4 - 4c_3^2 - 6c_2c_4)\varepsilon_n^6 + O(\varepsilon_n^7)]\end{aligned}\ , \qquad (13)$$

and

$$\begin{aligned}&5f(y_n) - f(\tilde{y}_n)\\ &= f'(\lambda)[-2\varepsilon_n + 2c_2\varepsilon_n^2 + 4(c_3 - 2c_2^2)\varepsilon_n^3 + 2(10c_2^3 + c_4 - 11c_2c_3)\varepsilon_n^4 + O(\varepsilon_n^5)]\end{aligned}\ , \qquad (14)$$

therefore, using (5), we get

$$\begin{aligned}&f'(x_n)[5f(y_n) - f(\tilde{y}_n)]\\ &= f'^2(\lambda)[-2\varepsilon_n - 2c_2\varepsilon_n^2 - 2(c_3 + 2c_2^2)\varepsilon_n^3 + 2(2c_2^3 - 3c_4 - 4c_2c_3)\varepsilon_n^4 + O(\varepsilon_n^5)]\end{aligned}\ . \qquad (15)$$

Now, dividing (13) by (15), we achieve

$$\frac{f(x_n)f(y_n)}{f'(x_n)\left[5f(y_n)-f(\tilde{y}_n)\right]} = -c_2\varepsilon_n^2+2(c_2^2-c_3)\varepsilon_n^3+(7c_2c_3-3c_4-3c_2^3)\varepsilon_n^4+O(\varepsilon_n^5) . \quad (16)$$

Finally, using (3), (16) and (7) we get the error relation:

$$\begin{aligned}\varepsilon_{n+1} &= x_{n+1}-\lambda \\ &= d_n+\frac{f(x_n)f(y_n)}{f'(x_n)\left[5f(y_n)-f(\tilde{y}_n)\right]} \\ &= c_2^3\varepsilon_n^4+O(\varepsilon_n^5),\end{aligned}$$

this implies that the method defined by (3) has a fourth-order.

Now, we construct a new fifth-order iterative method as follows

$$\begin{aligned}&y_n = x_n - f(x_n)/f'(x_n),\ \tilde{y}_n = x_n + f(x_n)/f'(x_n), \\ &x_{n+1} = y_n + \frac{2f^2(x_n)f(y_n)}{f'(x_n)\left[f(x_n)(5f(y_n)-f(\tilde{y}_n))+2f^2(y_n)\right]},\ n=0,1,2,\ldots\end{aligned} . \quad (17)$$

Theorem 2 Let λ be a simple zero of sufficiently differentiable function $f:I\subseteq R\rightarrow R$ for an open interval I. If x_0 is sufficiently close to λ, then the method defined by (17) is of fifth-order and satisfies the error equation

$$\varepsilon_{n+1} = 2c_2(c_2^3+c_4-c_2c_3)\varepsilon_n^5+O(\varepsilon_n^6)$$

Proof.

From (13), (14) and using (4), we have

$$\begin{aligned}&f^2(x_n)f(y_n) \\ &= f'^3(\lambda)[c_2\varepsilon_n^4+2c_3\varepsilon_n^5+(2c_2^3+3c_4-c_2c_3)\varepsilon_n^6 \\ &\quad +(10c_2^2c_3+4c_5-4c_2^4-2c_3^2-2c_2c_4)\varepsilon_n^7+O(\varepsilon_n^8)]\end{aligned} , \quad (18)$$

and

$$\begin{aligned}&f(x_n)\left[5f(y_n)-f(\tilde{y}_n)\right] \\ &= f'^2(\lambda)[-2\varepsilon_n^2+2(c_3-3c_2^2)\varepsilon_n^4+4c_2(3c_2^2-4c_3)\varepsilon_n^5+O(\varepsilon_n^6)\end{aligned} . \quad (19)$$

By (11), we obtain

$$f^2(y_n) = f'^2(\lambda)[c_2^2\varepsilon_n^4+4c_2(c_3-c_2^2)\varepsilon_n^5+(14c_2^4+6c_2c_4-22c_2^2c_3+4c_3^2)\varepsilon_n^6+O(\varepsilon_n^7)] , \quad (20)$$

and hence

$$\begin{aligned}&f(x_n)\left[5f(y_n)-f(\tilde{y}_n)\right]+2f^2(y_n) \\ &= f'^2(\lambda)[-2\varepsilon_n^2+2(c_3-2c_2^2)\varepsilon_n^4+4c_2(c_2^2-2c_3)\varepsilon_n^5+O(\varepsilon_n^6)\end{aligned} . \quad (21)$$

From (5) and (21), we attain

$$\begin{aligned}&f'(x_n)\left[f(x_n)\left(5f(y_n)-f(\tilde{y}_n)\right)+f^2(y_n)\right] \\ &= f'^3(\lambda)[-2\varepsilon_n^2-4c_2\varepsilon_n^3-4(c_2^2+c_3)\varepsilon_n^4-4(c_2^3+c_2c_3+2c_4)\varepsilon_n^5+O(\varepsilon_n^6)\end{aligned} . \quad (22)$$

Now, from (18) and (22), we obtain

$$\frac{2f^{2}(x_n)f(y_n)}{f'(x_n)\left[f(x_n)(5f(y_n)-f(\tilde{y}_n))+2f^{2}(y_n)\right]}$$
$$=-c_2\varepsilon_n^2+2(c_2^2-c_3)\varepsilon_n^3+(7c_2c_3-3c_4-4c_2^3)\varepsilon_n^4 \quad , \tag{23}$$
$$+2(6c_2c_4+5c_2^4+3c_3^2-11c_2^2c_3-2c_5)\varepsilon_n^5+O(\varepsilon_n^6)$$

and hence, by (17) and (23), we attain the error relation:

$$\varepsilon_{n+1}=x_{n+1}-\lambda$$
$$=d_n+\frac{2f^{2}(x_n)f(y_n)}{f'(x_n)\left[f(x_n)(5f(y_n)-f(\tilde{y}_n))+2f^{2}(y_n)\right]} \, ,$$
$$=2c_2(c_2^3+c_4-c_2c_3)\varepsilon_n^5+O(\varepsilon_n^6)$$

this certifies the method established by (17) has a fifth order.

Finally, we construct new seventh-order iterative method (7thOIM) as follows

$$y_n=x_n-f(x_n)/f'(x_n),$$
$$\hat{y}_n=y_n+\zeta_n,\ \breve{y}_n=y_n-\zeta_n \quad , \tag{24}$$
$$x_{n+1}=\hat{y}_n+\frac{2f^{2}(x_n)f(\hat{y}_n)\zeta_n}{4f^{2}(y_n)f'(x_n)\zeta_n-f^{2}(x_n)(f(\hat{y}_n)-f(\breve{y}_n))},\quad n=0,1,2,\ldots$$

where $\zeta_n=\dfrac{f(x_n)f(y_n)}{f'(x_n)\left[2f(y_n)-f(x_n)\right]}$.

Theorem 3 Let λ be a simple zero of sufficiently differentiable function $f: I \subseteq R \to R$ for an open interval I. If x_0 is sufficiently close to λ, then the method defined by (24) is of seventh-order and satisfies the error equation

$$\varepsilon_{n+1}=2c_2(2c_2-c_3-c_2^2)(2c_2^3-15c_2c_3-c_4)\varepsilon_n^7+O(\varepsilon_n^8)$$

Proof.

From (4) and (11), we have

$$2f(y_n)-f(x_n)$$
$$=f'(\lambda)[-\varepsilon_n+c_2\varepsilon_n^2+(3c_3-4c_2^2)\varepsilon_n^3+(10c_2^3+5c_4-14c_2c_3)\varepsilon_n^4 \quad , \tag{25}$$
$$+(48c_2^2c_3+7c_5-24c_2^4-12c_3^2-20c_2c_4)\varepsilon_n^5+O(\varepsilon_n^6)]$$

so by (5), we get

$$f'(x_n)\left[2f(y_n)-f(x_n)\right]$$
$$=f'^2(\lambda)[-\varepsilon_n-c_2\varepsilon_n^2-2c_2^2\varepsilon_n^3+(2c_2^3+c_4-5c_2c_3)\varepsilon_n^4 \quad , \tag{26}$$
$$+(8c_2^2c_3+2c_5-4c_2^4-3c_3^2-6c_2c_4)\varepsilon_n^5+O(\varepsilon_n^6)]$$

and hence, dividing (13) by (26), we attain

$$\xi_n=\frac{f(x_n)f(y_n)}{f'(x_n)[2f(y_n)-f(x_n)]}$$
$$=-c_2\varepsilon_n^2-2(c_3-c_2^2)\varepsilon_n^3+(6c_2c_3+2c_2^2-5c_2^3-3c_4)\varepsilon_n^4 \quad . \tag{27}$$
$$+(6c_2^4+4c_3^2+8c_2c_4-4c_5-12c_2^2c_3-2c_2^3)\varepsilon_n^5+O(\varepsilon_n^6)$$

Now, using (24), (27) and (7) we obtain

$$\begin{aligned}\hat{d}_n &= \hat{y}_n - \lambda \\ &= d_n + \xi_n \\ &= c_2(2c_2 - c_3 - c_2^2)\varepsilon_n^4 + (8c_2^2c_3 - 2c_3^2 - 2c_2c_4 - 2c_2^4 - 2c_2^3)\varepsilon_n^5 + O(\varepsilon_n^6)\end{aligned} \quad , \qquad (28)$$

furthermore

$$\begin{aligned}\breve{d}_n &= \breve{y}_n - \lambda \\ &= d_n - \xi_n \\ &= 2c_2\varepsilon_n^2 + 4(c_3 - c_2^2)\varepsilon_n^3 + (9c_2^3 + 6c_4 - 13c_2c_3 - 2c_2^2)\varepsilon_n^4 \\ &+ (8c_5 + 32c_2^2c_3 - 10c_3^2 - 18c_2c_4 - 14c_2^4 + 2c_2^3)\varepsilon_n^5 + O(\varepsilon_n^6)\end{aligned} \quad . \qquad (29)$$

Using Taylor expansion of $f(\hat{y}_n)$ and $f(\tilde{y}_n)$ around λ, we have

$$f(\hat{y}_n) = f'(\lambda)[\hat{d}_n + c_2\hat{d}_n^2 + c_3\hat{d}_n^3 + O(\hat{d}_n^4)] , \qquad (30)$$

$$f(\breve{y}_n) = f'(\lambda)[\breve{d}_n + c_2\breve{d}_n^2 + c_3\breve{d}_n^3 + O(\breve{d}_n^4)] , \qquad (31)$$

and hence, by (28) and (29) we attain

$$\begin{aligned}& f(\hat{y}_n) \\ &= f'(\lambda)[(2c_2^2 - c_2c_3 - c_2^3)\varepsilon_n^4 + (8c_2^2c_3 - 2c_3^2 - 2c_2c_4 - 2c_2^4 - 2c_2^3)\varepsilon_n^5 + O(\varepsilon_n^6)]\end{aligned} \quad , \qquad (32)$$

and

$$\begin{aligned}& f(\breve{y}_n) \\ &= f'(\lambda)[2c_2\varepsilon_n^2 + 4(c_3 - c_2^2)\varepsilon_n^3 + (13c_2^3 + 6c_4 - 13c_2c_3 - 2c_2^2)\varepsilon_n^4 \\ &\quad + (8c_5 + 48c_2^2c_3 - 10c_3^2 - 18c_2c_4 - 30c_2^4 + 2c_2^3)\varepsilon_n^5 + O(\varepsilon_n^6)]\end{aligned} \quad . \qquad (33)$$

Now, using (4), (32) and (33) we get

$$f^2(x_n) = f'^2(\lambda)[\varepsilon_n^2 + 2c_2\varepsilon_n^3 + (c_2^2 + 2c_3)\varepsilon_n^4 + (2c_4 + 2c_2c_3)\varepsilon_n^5 + O(\varepsilon_n^6)] , \qquad (34)$$

and

$$\begin{aligned}& f(\hat{y}_n) - f(\breve{y}_n) \\ &= f'(\lambda)[-2c_2\varepsilon_n^2 + 4(c_2^2 - c_3)\varepsilon_n^3 + (4c_2^2 + 12c_2c_3 - 14c_2^3 - 6c_4)\varepsilon_n^4 \\ &\quad + (28c_2^4 - 40c_2^2c_3 + 8c_3^2 + 16c_2c_4 - 4c_2^3 - 8c_5)\varepsilon_n^5 + O(\varepsilon_n^6)]\end{aligned} \quad , \qquad (35)$$

and hence, we get

$$\begin{aligned}& f^2(x_n)[f(\hat{y}_n) - f(\breve{y}_n)] \\ &= f'^3(\lambda)[-2c_2\varepsilon_n^4 - 4c_3\varepsilon_n^5 + (4c_2^2 - 8c_2^3 - 6c_4)\varepsilon_n^6 \\ &\quad + (4c_2^4 - 12c_2^2c_3 + 4c_2c_4 + 4c_2^3 - 8c_5)\varepsilon_n^7 + O(\varepsilon_n^8)]\end{aligned} \quad . \qquad (36)$$

From (20) and (27), we achieve

$$f^2(y_n)\xi_n = f'^2(\lambda)[-c_2^3\varepsilon_n^6 + 6c_2^2(c_2^2 - c_3)\varepsilon_n^7 + O(\varepsilon_n^8)] , \qquad (37)$$

therefore, by (5), we get

$$f^2(y_n)f'(x_n)\zeta_n = f'^3(\lambda)[-c_2^3\varepsilon_n^6 + 2c_2^2(2c_2^2 - 3c_3)\varepsilon_n^7 + O(\varepsilon_n^8)] . \qquad (38)$$

Now, from (36) and (38), we obtain

$$\begin{aligned}& 4f^2(y_n)f'(x_n)\zeta_n - f^2(x_n)[f(\hat{y}_n) - f(\breve{y}_n)] \\ &= f'^3(\lambda)[2c_2\varepsilon_n^4 + 4c_3\varepsilon_n^5 + (4c_2^3 + 6c_4 - 4c_2^2)\varepsilon_n^6 \\ &\quad + (12c_2^4 - 12c_2^2c_3 - 4c_2c_4 - 4c_2^3 + 8c_5)\varepsilon_n^7 + O(\varepsilon_n^8)]\end{aligned} \quad . \qquad (39)$$

Using (27) and (34), we get

$$f^2(x_n)\xi_n = f'^2(\lambda)[-c_2\varepsilon_n^4 - 2c_3\varepsilon_n^5 + (2c_2^2 - 2c_2^3 - 3c_4)\varepsilon_n^6 + (2c_2^3 - 2c_2^4 - 24c_2^2c_3 - 4c_5)\varepsilon_n^7 + O(\varepsilon_n^8)], \quad (40)$$

and hence, by applying (30), we attain

$$f(\hat{y}_n)f^2(x_n)\xi_n = f'^2(\lambda)[-c_2\varepsilon_n^4\hat{d}_n - 2c_3\varepsilon_n^5\hat{d}_n + (2c_2^2 - 2c_2^3 - 3c_4)\varepsilon_n^6\hat{d}_n + (2c_2^3 - 2c_2^4 - 24c_2^2c_3 - 4c_5)\varepsilon_n^7\hat{d}_n + O(\varepsilon_n^{12})], \quad (41)$$

therefore, using (39) and (41), we get

$$\frac{2f(\hat{y}_n)f^2(x_n)\xi_n}{4f^2(y_n)f'(x_n)\zeta_n - f^2(x_n)[f(\hat{y}_n) - f(\breve{y}_n)]} = -\hat{d}_n + 2(2c_2^3 - 15c_2c_3 - c_4)\varepsilon_n^3\hat{d}_n + O(\varepsilon_n^8). \quad (42)$$

Finally, using (24), (42) and (28) we get the error relation:

$$\begin{aligned}\varepsilon_{n+1} &= x_{n+1} - \lambda \\ &= \hat{d}_n + \frac{2f(\hat{y}_n)f^2(x_n)\xi_n}{4f^2(y_n)f'(x_n)\zeta_n - f^2(x_n)[f(\hat{y}_n) - f(\breve{y}_n)]} \\ &= 2c_2(2c_2 - c_3 - c_2^2)(2c_2^3 - 15c_2c_3 - c_4)\varepsilon_n^7 + O(\varepsilon_n^8)\end{aligned}.$$

This validates that the method defined by (24) is of seventh-order.

For every iteration, 4 function evaluations are required for both fourth order iterative method (4thOIM), (3), and fifth order iterative method (5thOIM), (17); 3 evaluations of f and one evaluation of f', while seventh order iterative method (7thOIM), (24), requires 5 function evaluation, 4 evaluations of f and one evaluation of f'. The method 4thOIM exhibits an efficiency index of $4^{1/4} \approx 1.4142$ as well as Newton's method (NM), (1). The method, 5thOIM demonstrates an efficiency index of $5^{1/4} \approx 1.4953$, and the method 7thOIM displays an efficiency index of $7^{1/5} \approx 1.4757$. These efficiency indexes outperform Newton's method (NM), (1) with an efficiency index of $2^{1/2} \approx 1.4142$, as well as the tenth order (TO) method [11], with efficiency index $10^{1/6} \approx 1.4677$.

3. Numerical Examples and Conclusion:

In this particular section, we deploy the recently introduced methods 4thOIM, 5thOIM, and 7thOIM defined by equations (3), (17) and (24) respectively. The purpose is to address nonlinear equations and compare the outcomes with Newton's method (NM) and the method (TO) [11].

The functions utilized in this context are listed below, [19]:

$$\begin{aligned}
f_1(x) &= x^3 + 4x^2 - 10, \ \lambda = 1.3652300134140968 8791373,\\
f_2(x) &= x^5 + x^4 + 4x^2 - 20, \ \lambda = 1.4662790738647226 7070587,\\
f_3(x) &= e^{x^2+7x-30} - 1, \ \lambda = 3,\\
f_4(x) &= (\sin x)^2 - x^2 + 1, \ \lambda = 1.4044916482153411 1524670,\\
f_5(x) &= e^x \sin x + \ln(x^2 + 1), \ \lambda = 0,\\
f_6(x) &= x^3 - \sin^2 x + 3\cos x + 5, \ \lambda = -1.5826870457520698 6540081,\\
f_7(x) &= x^3 - e^{-x}, \ \lambda = 0.7728829591492101 24749629.
\end{aligned}$$

The numerical demonstrations were executed in MatlabR2017b, utilizing 200 digits of floating-point precision and variable precision arithmetic. For the 7 functions mentioned above, we determined the solution for each test function with two different initial guesses x_0. The iterative procedures concluded when the error, $|x_{n+1} - x_n| + |f(x_n)|$, fell below 10^{-200}.

The number of iterations (IT) necessary under the condition $|x_{n+1} - x_n| + |f(x_n)| < 10^{-200}$ is presented in Table 1, while Table 2 illustrates the computational order ρ for all the examples considered.

Table 1 shows numerical outcomes suggesting that the presented techniques, denoted as 5thOIM and 7thOIM, achieve quicker convergence when contrasted with Newton's method (NM). They also involve a reduced number of iterations, highlighting the enhanced convergence efficiency of the new methods 4thOIM, 5thOIM and 7thOIM. Also, in Table 1, the method 4thOIM necessitate fewer iterations than Newton's method (NM).

The computational order of the four methods, NM (1), 4thOIM (3), 5thOIM (17), and 7thOIM (24), is presented in Table 2. The numerical findings in Table 2 affirm that the proposed methods uphold the theoretical results outlined in Section 2.

In summary, the conclusion drawn is that the recently introduced iterative methods, 5thOIM (17), and 7thOIM (24), detailed in this paper hold their own against other proficient equation solvers, like Newton's method (NM),(1) and the tenth-order method (TO). The efficiency indexes of 1.4953, 1.4757, 1.4142, and 1.4677 for methods 5thOIM, 7thOIM, NM, and (TO) respectively, serve as indicators of their performance.

Table 1. Numerical results for different methods with stopping criterium $|x_{n+1}-x_n|+|f(x_n)|<10^{-200}$

		The number of iterations (IT)				
$f(x)$	x_0	**NM(Eq.1)**	**TO([11])**	**4thOIM**	**5thOIM**	**7thOIM**
f_1	1.5	9	2	4	4	3
	1	10	3	5	4	3
f_2	1.2	10	3	5	4	4
	2	11	3	5	4	4
f_3	3.5	17	5	8	7	6
	4	24	8	12	10	8
f_4	1.6	10	3	4	4	3
	2.5	11	3	5	4	4
f_5	0.5	11	4	5	5	4
	2	11	4	6	5	4
f_6	-1	10	3	5	4	4
	-3	11	3	5	4	4
f_7	0	11	3	5	4	4
	1.5	11	3	5	4	4

Table 2. Numerical results for different methods with stopping criterium $|x_{n+1}-x_n|+|f(x_n)|<10^{-200}$

		NM(Eq.1)	**TO([11])**	**4thOIM**	**5thOIM**	**7thOIM**
$f(x)$	x_0	(IT, ρ)	(IT, ρ)	(IT, ρ)	(IT, ρ)	(IT, ρ)
f_1	1.5	(9, 2.0020)	(2, __)	(4, 4.0491)	(4, 5.0369)	(3, 7.1937)
	1	(10, 2.0018)	(3, 10.3010)	(5, 4.0213)	(4, 5.0707)	(3, 7.3794)
f_2	1.2	(10, 1.9999)	(3, 9.4298)	(5, 3.9992)	(4, 4.9893)	(4, 7.0018)
	2	(11, 2.0000)	(3, 9.5343)	(5, 3.9989)	(4, 4.9868)	(4, 7.0024)
f_3	3.5	(17,1.9957)	(5, 7.8713)	(8, 3.8494)	(7, 4.7633)	(6, 6.7199)
	4	(24, 1.9947)	(8, 9.3547)	(12, 3.8531)	(10,4.6235)	(8, 6.1657)
f_4	1.6	(10, 1.8269)	(3, 9.9212)	(4, 4.0227)	(4, 5.0162)	(3, 7.0302)
	2.5	(11, 2.0006)	(3, 9.7953)	(5, 4.0132)	(4, 5.0432)	(4, 7.0123)
f_5	0.5	(11, 1.9980)	(4, 9.8126)	(5, 3.9537)	(5, 4.9578)	(4, 6.8111
	2	(11, 1.9983)	(4, 9.6265)	(6, 3.9783)	(5, 4.9071)	(4, 6.7610)

f_6	-1	(10, 2.0029)	(3, 10.8454)	(5, 4.0341)	(4, 5.1177)	(4, 7.1021)
	-3	(11, 2.0022)	(3, 10.7783)	(5, 4.0534)	(4, 5.1300)	(4, 7.1275)
f_7	0	(11, 2.0002)	(3, 9.6018)	(5, 4.0044)	(4, 5.0223)	(4, 7.0212)
	1.5	(11, 2.0002)	(3, 9.6159)	(5, 4.0048)	(4, 5.0237)	(4, 7.0197)

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ثلاث طرق تكرارية من الرتبة الرابعة والرتبة الخامسة والرتبة السابعة لإيجاد جذور المعادلات غير الخطية

حسن محمد سعيد باوزير

الملخص: في هذه الورقة قدمت ثلاث طرق لإيجاد الحلول العددية للمعادلات غير الخطية. لقد تم برهان أن الطرق تبدي رتب التقارب: أربعة وخمسة وسبعة. التحليل يظهر أن كفاءة الطرق تتفوق على بعض الطرق الحديثة. تم تقييم أداء الطرق المقدمة من خلال أمثلة عددية، بالإضافة أن رتب التقارب تم تأكيدها خلال هذه الأمثلة.

كلمات مفتاحية: معادلة غير خطية، طريقة تكرارية، طريقة نيوتن، رتبة التقارب.