Hadhramout University Journal of Natural & Applied Science Vol 22 No 2 (2025) 119 – 124

NAS journal Hadhramout University

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Hadhramout University Journal of Natural & Applied Science

Article

Digital Object Identifier:
Received 23 June 2025,
Accepted 20 July 2025,
Available online 25 January 2026

ON (p,q)-BESSEL FUNCTION OF TWO VARIABLES

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Abstract: The Bessel function is probably the best-known special function, within pure and applied mathematics. The principal object of the present paper is to construct new analogy definition of the (p,q)-Bessel function of two variables, using the generating function method. This study shows a class of Bessel function with the help of the generating function. Some properties of this function and explicit representations of difference equations and recurrence relations are obtained.

Keywords: Bessel function; Two-variable special functions; Generating Function; Difference Equations; Special functions

1. INTRODUCTION AND NOTATIONS

The theory (p,q)-calculus has recently been applied in many branches of pure and applied mathematics, physics and engineering, such as biology, electrochemistry, economics, engineering, probability theory, statistics, statistical sciences, quantum theory, number theory and statistical mechanics, etc. The (p,q)-special functions have important roles in many areas of mathematical physics and mathematics (see, for example [4,5,10,11,14,18,20,21]).

In this section, we will give a summary of the definitions and mathematical notations required in this paper for the convenience of the reader.

The Bessel's function of first kind and r order defined by [17]

$$J_r(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2n+r} \tag{1.1}$$

The explicit representation of the q-analogue Bessel function of one variable is given by [6,7]

$$J_r(x;q)=\sum_{n=0}^{\infty}\frac{(-1)^n}{[n]_q![r+n]_q!}\left(\frac{x}{2}\right)^{2n+r} \tag{1.2}$$

where $\lim_{q\to 1} J_r\big((1-q)x;q\big)=J_r(x)$.

Mahmoud [12] introduced and studied the generalized q-Bessel function as follow:

$$J_n(x,a;q)=\frac{\left({}^{x}/_{2}\right)^n}{(q;q)_n}\sum_{r=0}^{\infty}(-1)^{r(a+1)}\frac{q^{\frac{ar}{2}(r+n)}}{(q^{n+1};q)_n}\frac{\left({}^{x^2}/_{4}\right)^r}{(q;q)_r} \tag{1.3}$$

which converges absolutely for all x when $a\in Z^+$ and for $|x|<2$ if $a=0$.

Shehata [23] presented the (p,q)-Bessel function of one variable defined by

$$J_n^{(1)}(x;p,q)=\sum_{r=0}^{\infty}(-1)^r\frac{p^{\binom{n+r}{2}+\binom{r}{2}}}{[r]_{p,q}![n+r]_{p,q}!}\left(\frac{x}{2}\right)^{n+2r} \tag{1.4}$$

Bessel's functions of two variables are defined by the following series [1]:

$$J_{r,s}(x,y)=\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}}{n!m!\Gamma(r+m+1)\Gamma(s+n+1)}\left(\frac{x}{2}\right)^{2m+r}\left(\frac{yp(x)}{2}\right)^{2n+s} \tag{1.5}$$

and

$$J_{r,s}(x,y)=\frac{\left(\frac{x}{2}\right)^r\left(\frac{yp(x)}{2}\right)^s}{\Gamma(r+1)\Gamma(s+1)}\ {}_0F_1\left(-;r+1;-\frac{x^2}{4}\right)\ {}_0F_1\left(-;s+1;-\frac{y^2p^2(x)}{4}\right) \tag{1.6}$$

where r and s are integers.

Tenguria and Sharma [24] presented a study of the advanced q-Bessel function of two variables is given by

$$J_{r,s}(x,y;q)=\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}}{[m]_q![r+m]_q![n]_q![s+n]_q!}\left(\frac{x}{2}\right)^{2m+r}\left(\frac{yp(x)}{2}\right)^{2n+s} \tag{1.7}$$

Thus, Alsarahi [2] introduced and studied of the generalized q-analogue Bessel matrix polynomials of two variables

$$J_{r,s}(x,y,a,A,B;q)=\frac{\left(\frac{x\sqrt{2A}}{2}\right)^r\left(\frac{Byp(x)}{2}\right)^s}{(q;q)_r(q;q)_s}$$
$$\times\sum_{m,n=0}^{\infty}\frac{(-1)^{(m+n)(a+1)}q^{a/2[m(m+r)+n(n+s)]}}{(q^{1+r};q)_m(q;q)_m(q^{1+s};q)_n(q;q)_n}\left(\frac{x\sqrt{2A}}{2}\right)^{2m}\left(\frac{Byp(x)}{2}\right)^{2n} \quad (1.8)$$

The q-shifted factorial be defined by [18,19,20]

$$[\alpha]_q=\frac{1-q^\alpha}{1-q},\ 0<|q|<1;q\in C-\{1\};\ \ \alpha\in C \quad (1.9)$$

where $\lim_{q\to1}[\alpha]_q=\lim_{q\to1}\frac{1-q^\alpha}{1-q}=\alpha.$

The q-analogue of $n!$ is then defined by

$$[n]_q!=\begin{cases}1 & ,\quad n=0\\ [n]_q[n-1]_q\dots[2]_q[1]_q, & n\in N\end{cases} \quad (1.10)$$

The (p,q)-number is denoted by $[\beta]_{p,q}$ and is given by

$$[\beta]_{p,q}=\frac{p^\beta-q^\beta}{p-q},\ 0<|q|<|p|\le1;\ \ p,q,\beta\in C \quad (1.11)$$

The (p,q)-number and (p,q)-factorial are given as follow. (see [8,9,13])

$$[k]_{p,q}=\begin{cases}\frac{p^k-q^k}{p-q}, & k\in N\\ 0, & k=0\end{cases},$$

where

$$\lim_{p\to1}[k]_{p,q}=[k]_q=\frac{1-q^k}{1-q},q\neq1$$

The (p,q)-factorial is denoted by $[n]_{p,q}!$ is defined by

$$[n]_{p,q}!=\prod_{k=1}^n[k]_{p,q}=\frac{((p,q);(p,q))_n}{(p-q)^n},n\ge1;\ \ [0]_{p,q}!=1. \quad (1.12)$$

Let us introduce (p,q)-binomial coefficients is given by

$$\begin{bmatrix}n\\r\end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[n-r]_{p,q}![r]_{p,q}!},\ \ 0\le r\le n,\ \ r,n\in N \quad (1.13)$$

The (p,q)-exponential function $e_{p,q}(x)$ is defined by (see [18,23])

$$e_{p,q}(x)=\sum_{k=0}^\infty p^{\binom{k}{2}}\frac{x^k}{[k]_{p,q}!}. \quad (1.14)$$

The (p,q)-derivative operator for any function f is defined as follows (see [7,15,16])

$$D_{p,q}f(x)=\frac{f(px)-f(qx)}{(p-q)x},\quad x\neq0 \quad (1.15)$$

and $\left((D_{p,q}f)(0)=f'(0)\right).$

Thus, the operator $D_{p,q}$ satisfies the following relation (see [18])

$$D_{p,q}e_{p,q}(\mu x)=\mu e_{p,q}(\mu px),\ \mu\in C \quad (1.16)$$

where μ is a complex number.

It happens clearly that $D_{p,q}x^n=[n]_{p,q}x^{n-1}$.

Note also that for $p=1$ in (1.15), the $(p,\ q)$-derivative reduces to the q-derivative which is defined by

$$D_qf(x)=\frac{f(x)-f(qx)}{(1-q)x},\quad x\neq0$$

The (p,q)-derivative operator satisfy the following product rule (see [10, 15,16]):

$$D_{p,q}\big(f(x).h(x)\big)=f(px)D_{p,q}h(x)+h(qx)D_{p,q}f(x) \quad (1.17)$$

The primary goal of this work is to introduce and study the (p,q)-Bessel function of two variables and some of its properties, (p,q)-difference equations and recurrence relations are obtained.

2. The (p,q)-Bessel Function of Two Variables

The generating function of the (p,q)-Bessel function of two variables, denoted by $J_{m,n}(x,y;p,q)$, is define by the following relation:

$$e_{p,q}\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right].e_{p,q}\left[\frac{yf(x)}{2}\left(w-\frac{1}{w}\right)\right]=\sum_{m,n=-\infty}^{\infty}J_{m,n}(x,y;p,q)\,t^mw^n \quad (2.1)$$

where $x,y\in R,\ t,w\in\mathbb{C},\ \ t,w\neq0, 0<|q|<|p|\le1,\ q,p\in C.$

Now, by using the above generating function (2.1), we will deduce that the (p,q)-Bessel function of two variables $J_{m,n}(x,y;p,q)$ in the following theorem:

Theorem 2.1. Suppose that $x,y\in R,x,y>0,f(x)>0,t,w\in\mathbb{C},\ \ t,w\neq0,\ \ 0<|q|<|p|\le1,\ q,p\in C$ then the (p,q)-Bessel function of two variables $J_{m,n}(x,y;p,q)$ has the following representation:

$$J_{m,n}(x,y;p,q)=\left(\frac{x}{2}\right)^m\left(\frac{yf(x)}{2}\right)^n$$
$$\times\sum_{r,s=0}^\infty(-1)^{r+s}\frac{p^{\binom{m+r}{2}+\binom{r}{2}+\binom{n+s}{2}+\binom{s}{2}}}{[r]_{p,q}![s]_{p,q}!\Gamma_{p,q}(m+r+1)\Gamma_{p,q}(n+s+1)}\left(\frac{x}{2}\right)^{2r}\left(\frac{yf(x)}{2}\right)^{2s} \quad (2.2)$$

Proof. Let us denoted the left hand sides of (2.1) by H, we have

$$H=e_{p,q}\left[\frac{xt}{2}\right]e_{p,q}\left[-\frac{x}{2t}\right]e_{p,q}\left[\frac{yf(x)w}{2}\right]e_{p,q}\left[-\frac{yf(x)}{2w}\right]$$

Appling relation (1.14), we obtain

$$H=\sum_{m=0}^\infty\sum_{r=0}^\infty(-1)^r\frac{p^{\binom{m}{2}+\binom{r}{2}}}{[m]_{p,q}![r]_{p,q}!}\left(\frac{x}{2}\right)^{m+r}t^{m-r}$$
$$\times\sum_{n=0}^\infty\sum_{s=0}^\infty(-1)^s\frac{p^{\binom{n}{2}+\binom{s}{2}}}{[n]_{p,q}![s]_{p,q}!}\left(\frac{yf(x)}{2}\right)^{n+s}w^{n-s} \quad (2.3)$$

Replace m and n by $m+r$ and $n+s$ respectively in the right hand side of equation (2.3), we get

$$H=\sum_{m,n=-\infty}^\infty\left(\frac{x}{2}\right)^m\left(\frac{yf(x)}{2}\right)^n$$
$$\times\sum_{r,s=0}^\infty(-1)^{r+s}\frac{p^{\binom{m+r}{2}+\binom{r}{2}+\binom{n+s}{2}+\binom{s}{2}}}{[m+r]_{p,q}![r]_{p,q}![n+s]_{p,q}![s]_{p,q}!}\left(\frac{x}{2}\right)^{2r}\left(\frac{yf(x)}{2}\right)^{2s}t^mw^n \quad (2.4)$$

Which on using definition (2.1), we get the required result (2.2).

Remark 2.1. In result (2.2), we can see that

$$\lim_{\substack{p\to1\\q\to1}}J_{m,n}(x,0;p,q)=J_{m,n}(x),$$
$$\lim_{p\to1}J_{m,n}(x,0;p,q)=J_{m,n}(x;q),$$
$$J_{m,n}(x,0;p,q)=J_{m,n}(x;p,q),$$
$$\lim_{\substack{p\to1\\q\to1}}J_{m,n}(x,y;p,q)=J_{m,n}(x,y),$$
$$\lim_{p\to1}J_{m,n}(x,y;p,q)=J_{m,n}(x,y;q),$$

which obtain equations (1.1), (1.2), (1.4). (1.5) and (1.7) respectively.

Remark 2.2. By using the relation (1.12), the series expansion of the (p,q)-Bessel function $J_{m,n}(x,y;p,q)$ is given as

$$J_{m,n}(x,y;p,q)=\frac{\left(\frac{x}{2}\right)^m\left(\frac{yf(x)}{2}\right)^n}{\big((p,q);(p,q)\big)_n\big((p,q);(p,q)\big)_m}$$

$$\times \sum_{r,s=0}^{\infty} (-1)^{r+s} \frac{p^{\binom{m+r}{2}+\binom{r}{2}+\binom{n+s}{2}+\binom{s}{2}}}{\left((p^{m+1},q^{m+1});(p,q)\right)_r \left((p^{n+1},q^{n+1});(p,q)\right)_s}$$

$$\times \frac{1}{((p,q);(p,q))_r ((p,q);(p,q))_s} \left(\frac{(p-q)x}{2}\right)^{2r} \left(\frac{(p-q)yf(x)}{2}\right)^{2s} \quad (2.5)$$

Corollary 2.1. If m,n be integer, then $J_{m,n}(x,y;p,q)$ satisfies the following relations:

$$J_{-m,n}(x,y;p,q) = (-1)^m J_{m,n}(x,y;p,q) \quad (2.6)$$

$$J_{m,-n}(x,y;p,q) = (-1)^n J_{m,n}(x,y;p,q) \quad (2.7)$$

$$J_{-m,-n}(x,y;p,q) = (-1)^{m+n} J_{m,n}(x,y;p,q) \quad (2.8)$$

Proof. From (2.2), we get

$$J_{-m,n}(x,y;p,q) = \left(\frac{x}{2}\right)^{-m} \left(\frac{yf(x)}{2}\right)^n$$

$$\times \sum_{r=m}^{\infty} \sum_{s=0}^{\infty} (-1)^{r+s} \frac{p^{\binom{r-m}{2}+\binom{r}{2}+\binom{n+s}{2}+\binom{s}{2}}}{[r-m]_{p,q}![r]_{p,q}![n+s]_{p,q}![s]_{p,q}!} \left(\frac{x}{2}\right)^{2r} \left(\frac{yf(x)}{2}\right)^{2s} \quad (2.9)$$

substituting r by $r+m$ in the right hand side of (2.9) , we obtain

$$J_{-m,n}(x,y;p,q) = \left(\frac{x}{2}\right)^{m} \left(\frac{yf(x)}{2}\right)^n$$

$$\times \sum_{r,s=0}^{\infty} (-1)^{r+m+s} \frac{p^{\binom{r}{2}+\binom{r+m}{2}+\binom{n+s}{2}+\binom{s}{2}}}{[r]_{p,q}![r+m]_{p,q}![n+s]_{p,q}![s]_{p,q}!} \left(\frac{x}{2}\right)^{2r} \left(\frac{yf(x)}{2}\right)^{2s} \quad (2.10)$$

Hence

$$J_{-m,n}(x,y;p,q) = (-1)^m \left(\frac{x}{2}\right)^{m} \left(\frac{yf(x)}{2}\right)^n$$

$$\times \sum_{r,s=0}^{\infty} (-1)^{r+s} \frac{p^{\binom{r}{2}+\binom{r+m}{2}+\binom{n+s}{2}+\binom{s}{2}}}{[r]_{p,q}!\,[r+m]_{p,q}!\,[n+s]_{p,q}!\,[s]_{p,q}!} \left(\frac{x}{2}\right)^{2r} \left(\frac{yf(x)}{2}\right)^{2s}$$

which is the required relation (2.6).

Similarly, we can show that the relations (2.7) and (2.8).

Corollary 2.2. For the (p,q)-Bessel function $J_{m,n}(x,y;p,q)$ the following hold:

$$J_{m,n}(-x,y;p,q) = (-1)^m J_{m,n}(x,y;p,q) \quad (2.11)$$

$$J_{m,n}(x,-y;p,q) = (-1)^n J_{m,n}(x,y;p,q) \quad (2.12)$$

$$J_{m,n}(-x,-y;p,q) = (-1)^{m+n} J_{m,n}(x,y;p,q) \quad (2.13)$$

Proof. If $f(x)$ is even function and m,n,r,s any integers, then

$$J_{m,n}(-x,y;p,q) = \left(-\frac{x}{2}\right)^{m} \left(\frac{yf(-x)}{2}\right)^n$$

$$\times \sum_{r,s=0}^{\infty} \frac{(-1)^{r+s} p^{\binom{m+r}{2}+\binom{r}{2}+\binom{n+s}{2}+\binom{s}{2}}}{[r]_{p,q}!\,[s]_{p,q}!\,\Gamma_{p,q}(m+r+1)\Gamma_{p,q}(n+s+1)} \left(-\frac{x}{2}\right)^{2r} \left(\frac{yf(-x)}{2}\right)^{2s}$$

$$= (-)^m \left(\frac{x}{2}\right)^{m} \left(\frac{yf(x)}{2}\right)^n$$

$$\times \sum_{r,s=0}^{\infty} \frac{(-1)^{r+s} p^{\binom{m+r}{2}+\binom{r}{2}+\binom{n+s}{2}+\binom{s}{2}}}{[r]_{p,q}!\,[s]_{p,q}!\,\Gamma_{p,q}(m+r+1)\Gamma_{p,q}(n+s+1)} \left(\frac{x}{2}\right)^{2r} \left(\frac{yf(x)}{2}\right)^{2s}$$

which in view of (2.2) yields relation (2.11).

Similarly, in same way we can be proved the relations (2.11) and (2.13).

3. The (p,q)-difference Equations

Theorem (3.1). The (p,q)-Bessel functions $J_{m,n}(x,y;p,q)$ satisfy the following relations:

$$\frac{\partial_{(p,q)}}{\partial_{(p,q)}x} J_{m,n}(x,y;p,q) = \frac{yf'(x)}{2}\Big(J_{m,n-1}(px,py;p,q) - J_{m,n+1}(px,py;p,q)\Big)$$

$$+\frac{1}{2}\Big(J_{m-1,n}(px,qy;p,q) - J_{m+1,n}(px,qy;p,q)\Big) \quad (3.1)$$

and

$$\frac{\partial_{(p,q)}}{\partial_{(p,q)}y} J_{m,n}(x,y;p,q) = \frac{f(x)}{2}\Big(J_{m,n-1}(x,py;p,q) - J_{m,n+1}(x,py;p,q)\Big) \quad (3.2)$$

Proof. Differentiating (2.1) with respect to x and using (1.16) and (1.17), yields

$$\sum_{m,n=-\infty}^{\infty} \frac{\partial_{(p,q)}}{\partial_{(p,q)}x} J_{m,n}(x,y;p,q) t^m w^n$$

$$= \frac{yf'(x)}{2}\left(w-\frac{1}{w}\right) e_{p,q}\left[\frac{px}{2}\left(t-\frac{1}{t}\right)\right] \cdot e_{p,q}\left[\frac{pyf(x)}{2}\left(w-\frac{1}{w}\right)\right]$$

$$+\frac{1}{2}\left(t-\frac{1}{t}\right) e_{p,q}\left[\frac{px}{2}\left(t-\frac{1}{t}\right)\right] \cdot e_{p,q}\left[\frac{qyf(x)}{2}\left(w-\frac{1}{w}\right)\right]$$

$$= \frac{yf'(x)}{2}\left(w-\frac{1}{w}\right) e_{p,q}\left[\frac{pxt}{2}\right] e_{p,q}\left[-\frac{px}{2t}\right] e_{p,q}\left[\frac{pyf(x)w}{2}\right] e_{p,q}\left[-\frac{pyf(x)}{2w}\right]$$

$$+\frac{1}{2}\left(t-\frac{1}{t}\right) e_{p,q}\left[\frac{pxt}{2}\right] e_{p,q}\left[-\frac{px}{2t}\right] e_{p,q}\left[\frac{qyf(x)w}{2}\right] e_{p,q}\left[-\frac{qyf(x)}{2w}\right]$$

Using (1.14) in the above equation, we obtain

$$\sum_{m,n=-\infty}^{\infty} \frac{\partial_{(p,q)}}{\partial_{(p,q)}x} J_{m,n}(x,y;p,q) t^m w^n$$

$$= \frac{yf'(x)}{2}\left(w-\frac{1}{w}\right) \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} (-1)^r \frac{p^{\binom{m}{2}+\binom{r}{2}}}{[m]_{p,q}!\,[r]_{p,q}!} \left(\frac{px}{2}\right)^{m+r} t^{m-r}$$

$$\times \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} (-1)^s \frac{p^{\binom{n}{2}+\binom{s}{2}}}{[n]_{p,q}!\,[s]_{p,q}!} \left(\frac{pyf(x)}{2}\right)^{n+s} w^{n-s}$$

$$+\frac{1}{2}\left(t-\frac{1}{t}\right) \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} (-1)^r \frac{p^{\binom{m}{2}+\binom{r}{2}}}{[m]_{p,q}!\,[r]_{p,q}!} \left(\frac{px}{2}\right)^{m+r} t^{m-r}$$

$$\times \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} (-1)^s \frac{p^{\binom{n}{2}+\binom{s}{2}}}{[n]_{p,q}![s]_{p,q}!} \left(\frac{qyf(x)}{2}\right)^{n+s} w^{n-s}. \quad (3.3)$$

Replace m by $r+m$ and n by $s+n$ in the right hand side of (3.3), we get

$$\sum_{m,n=-\infty}^{\infty} \frac{\partial_{(p,q)}}{\partial_{(p,q)}x} J_{m,n}(x,y;p,q) t^m w^n$$

$$= \frac{yf'(x)}{2}\left(w-\frac{1}{w}\right) \sum_{m,n=-\infty}^{\infty}\sum_{r,s=0}^{\infty} \frac{(-1)^{r+s} p^{\binom{m+r}{2}+\binom{r}{2}+\binom{n+s}{2}+\binom{s}{2}}}{[m+r]_{p,q}!\,[r]_{p,q}!\,[n+s]_{p,q}!\,[s]_{p,q}!}$$

$$\times \left(\frac{px}{2}\right)^{m+2r} \left(\frac{pyf(x)}{2}\right)^{n+2s} t^m w^n$$

$$+\frac{1}{2}\left(t-\frac{1}{t}\right) \sum_{m,n=-\infty}^{\infty}\sum_{r,s=0}^{\infty} (-1)^{r+s} \frac{p^{\binom{m+r}{2}+\binom{r}{2}+\binom{n+s}{2}+\binom{s}{2}}}{[m+r]_{p,q}!\,[r]_{p,q}!\,[n+s]_{p,q}!\,[s]_{p,q}!}$$

$$\times \left(\frac{px}{2}\right)^{m+2r} \left(\frac{qyf(x)}{2}\right)^{n+2s} t^m w^n. \quad (3.4)$$

By equating the coefficients of $t^m w^n$ in (3.4), we get relation (3.1).

Also, differentiating (2.1) with respect to y and using (1.16), we find

$$\sum_{m,n=-\infty}^{\infty}\frac{\partial_{(p,q)}}{\partial_{(p,q)}y}J_{m,n}(x,y;p,q)t^m w^n = \frac{f(x)}{2}\left(w-\frac{1}{w}\right)e_{p,q}\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right].e_{p,q}\left[\frac{pyf(x)}{2}\left(w-\frac{1}{w}\right)\right] \quad (3.5)$$

Using relation (1.14) and replacing m by $r+m$ and n by $s+n$ in the equation (3.6), we obtain

$$\sum_{m,n=-\infty}^{\infty}\frac{\partial_{(p,q)}}{\partial_{(p,q)}y}J_{m,n}(x,y;p,q)t^m w^n = \frac{f(x)}{2}\left(w-\frac{1}{w}\right) \times \sum_{m,n=-\infty}^{\infty}\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}p^{\binom{m+r}{2}+\binom{r}{2}+\binom{n+s}{2}+\binom{s}{2}}}{[m+r]_{p,q}![r]_{p,q}![n+s]_{p,q}![s]_{p,q}!}\left(\frac{x}{2}\right)^{m+2r}\left(\frac{pyf(x)}{2}\right)^{n+2s}t^m w^n. \quad (3.6)$$

Equating the coefficients of $t^m w^n$ in (3.7), we get result (3.2).

4. Recurrence Relations

Theorem (4.1). The polynomials sequence $J_{m,n}(x,y;p,q)$ satisfies the next recurrence relations:

$$[m+1]_{p,q}J_{m+1,n}(x,y;p,q) = \frac{p^{\frac{m}{2}}x}{2q^{\frac{m}{2}}}J_{m,n}\left((pq)^{\frac{1}{2}}x,y;p,q\right)+\frac{x}{2t^2}J_{m,n}(px,y;p,q) \quad (4.1)$$

and

$$[n+1]_{p,q}J_{m,n+1}(x,y;p,q) = \frac{p^{\frac{n}{2}}yf(x)}{2q^{\frac{n}{2}}}J_{m,n}\left(x,(pq)^{\frac{1}{2}}y;p,q\right)+\frac{yf(x)}{2w^2}J_{m,n}(x,py;p,q) \quad (4.2)$$

Proof. Differentiating (2.1) with respect to t and using (1.16) and (1.17), we find

$$\sum_{m,n=-\infty}^{\infty}\frac{\partial_{(p,q)}}{\partial_{(p,q)}t}J_{m,n}(x,y;p,q)t^m w^n = \frac{x}{2}e_{p,q}\left[\frac{pxt}{2}\right]e_{p,q}\left[-\frac{qx}{2t}\right]e_{p,q}\left[\frac{yf(x)w}{2}\right]e_{p,q}\left[-\frac{yf(x)}{2w}\right] + \frac{x}{2t^2}e_{p,q}\left[\frac{pxt}{2}\right]e_{p,q}\left[-\frac{px}{2t}\right]e_{p,q}\left[\frac{yf(x)w}{2}\right]e_{p,q}\left[-\frac{yf(x)}{2w}\right]$$

By using relation (1.14), we obtain

$$\sum_{m,n=-\infty}^{\infty}[m]_{p,q}J_{m,n}(x,y;p,q)t^{m-1}w^n = \frac{x}{2}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}(-1)^r\frac{p^{\binom{m}{2}+\binom{r}{2}}q^r(p)^m}{[m]_{p,q}![r]_{p,q}!}\left(\frac{x}{2}\right)^{m+r}t^{m-r} \times \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s\frac{p^{\binom{n}{2}+\binom{s}{2}}}{[n]_{p,q}![s]_{p,q}!}\left(\frac{yf(x)}{2}\right)^{n+s}w^{n-s} + \frac{x}{2t^2}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}(-1)^r\frac{p^{\binom{m}{2}+\binom{r}{2}}}{[m]_{p,q}![r]_{p,q}!}\left(\frac{px}{2}\right)^{m+r}t^{m-r} \times \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s\frac{p^{\binom{n}{2}+\binom{s}{2}}}{[n]_{p,q}![s]_{p,q}!}\left(\frac{yf(x)}{2}\right)^{n+s}w^{n-s}. \quad (4.3)$$

Substituting m and n by $r+m$ and $s+n$ respectively in (4.3), we get

$$\sum_{m,n=-\infty}^{\infty}[m+1]_{p,q}J_{m+1,n}(x,y;p,q)t^m w^n = \frac{x}{2}\sum_{m,n=-\infty}^{\infty}\frac{p^{\frac{m}{2}}}{q^{\frac{m}{2}}} \times \sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}p^{\binom{m+r}{2}+\binom{r}{2}+\binom{n+s}{2}+\binom{s}{2}}}{[m+r]_{p,q}![r]_{p,q}![n+s]_{p,q}![s]_{p,q}!}\left(\frac{(pq)^{\frac{1}{2}}x}{2}\right)^{m+2r}\left(\frac{yf(x)}{2}\right)^{n+2s}t^m w^n$$

$$+\frac{x}{2t^2}\sum_{m,n=-\infty}^{\infty}\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}p^{\binom{m+r}{2}+\binom{r}{2}+\binom{n+s}{2}+\binom{s}{2}}}{[m+r]_{p,q}![r]_{p,q}![n+s]_{p,q}![s]_{p,q}!}\left(\frac{px}{2}\right)^{m+2r}\left(\frac{yf(x)}{2}\right)^{n+2s}t^m w^n$$

Comparing of both sides, we get the required relation (4.1).

Similarly way differentiating (2.1) with respect to w, we can prove the relation (4.2).

Theorem (4.2). For polynomial $J_{m,n}(x,y;p,q)$ holds

$$\frac{p^{\binom{m}{2}+\binom{n}{2}}}{[m]_{p,q}![n]_{p,q}!}\left(\frac{x}{2}\right)^m\left(\frac{yf(x)}{2}\right)^n = \sum_{r,s=0}^{\infty}\frac{p^{\binom{r}{2}+\binom{s}{2}}}{[r]_{p,q}![s]_{p,q}!}\left(\frac{x}{2}\right)^r\left(\frac{yf(x)}{2}\right)^s J_{m+r,n+s}(x,y;p,q) \quad (4.4)$$

Proof. The generating function of polynomial $J_{m,n}(x,y;p,q)$ we can written as

$$e_{p,q}\left[\frac{xt}{2}\right].e_{p,q}\left[\frac{yf(x)w}{2}\right] = e_{p,q}\left[\frac{x}{2t}\right].e_{p,q}\left[\frac{yf(x)}{2w}\right]\sum_{m,n=-\infty}^{\infty}J_{m,n}(x,y;p,q)\,t^m w^n.$$

From the relation (1.13)

$$\sum_{m,n=0}^{\infty}\frac{p^{\binom{m}{2}+\binom{n}{2}}}{[m]_{p,q}![n]_{p,q}!}\left(\frac{x}{2}\right)^m\left(\frac{yf(x)}{2}\right)^n t^m w^n = \sum_{r,s=0}^{\infty}\frac{p^{\binom{r}{2}+\binom{s}{2}}}{[r]_{p,q}![s]_{p,q}!}\left(\frac{x}{2}\right)^r\left(\frac{yf(x)}{2}\right)^s\sum_{m,n=0}^{\infty}J_{m+r,n+s}(x,y;p,q)\,t^m w^n \quad (4.5)$$

On equating of the coefficients of $t^m w^n$ in (4.5), we obtain the required relation (4.4).

5. Conclusion and Perspectives

In this work, we have introduced (p,q)-Bessel function of two variables or the twin-basic Bessel function, using the family of generating function method. We have seen some particular cases of (p,q)-Bessel function of two variables, which can a starting point. In a forthcoming works will be carried out in proposing modified forms (p,q)-Bessel functions in other fields of mathematical physics and engineering sciences and so on.

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دالة(p,q)– بيسل لمتغيرين

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الملخص: تُعد دالة بيسل من أشهر الدال الخاصة في الرياضيات البحتة والتطبيقية. يهدف هذا البحث إلى بناء تعريف تناظري جديد لدالة (p,q) –بيسل لمتغيرين، باستخدام طريقة دالة المولدة، حيث تظهر هذه الدراسة فئة من دالة (p,q)–بسل باستخدام الدال المولدة، كما تُستخلص بعض خصائص هذه الدالة، مثل التمثيلات الصريحة لمعادلات الفرق والعلاقات التكرارية.

الكلمات المفتاحية: دالة بيسل؛ الدوال الخاصة ذات المتغيرين؛ الدالة المولدة؛ المعادلات التفاضلية؛ الدوال الخاصة.