ON θ-$\mathcal{G}^S$-Closed Sets and θ-$\mathcal{G}^S$-Continuous Functlons

Amin Hamoud Saif* **Nahid Mohammed Al-showhati****

Abstract

In topological spaces, the class of θ-closed sets and θ-continuous function have introduced by Velicko and Fomin respectively. The purpose of this paper is to introduce and study these notions in grill topological spaces by giving the new classes of θ-$\mathcal{G}^S$-closed sets and θ-$\mathcal{G}^S$-continuous functions in grill topological space.
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Introduction:

Let $(X,\mathcal{T})$ be a topological space and $A \subseteq X$. Throughout this paper, the closure and the interior of A will be denoted by Cl(A) and Int(A), respectively. Fomin [4] introduced the concept of θ-continuous functions and Velicko [7] introduced the concept of θ-closed sets in topological spaces. Recall [4] that a function f: $(X,\mathcal{T}) \to (Y,P)$ of a topological space $(X,\mathcal{T})$ into a topological space (Y,P) is called θ-continuous function at $x \in X$ if for each open set V of f(x), there exists an open set U containing x such that $f(Cl(U)) \subseteq Cl(V)$. The function f is said to be θ-continuous if it is θ-continuous at each point in X. Recall [7] that a point $x \in X$ is called θ-cluster point of A if $Cl(U) \cap A \neq \phi$, for every open set U in X containing x. The set of all θ-cluster points of A is called the θ-closure set of A and is denoted by $Cl^{\theta}(A)$. A subset A of topological space is called θ-closed set in X if $Cl^{\theta}(A) = A$. The complement of θ-closed set in X is called θ-open set in X.

This paper is organized as follows. Section 2 explains some concepts and facts in grill topological spaces. In Section 3, we introduce the concept of $\mathcal{G}^S$-open set and some results about it. Section 4, is devoted to introduce and study classes of θ-$\mathcal{G}^S$-closed sets. Finally in section 5 we introduce and study classes of θ-$\mathcal{G}^S$-continuous functions.

Preliminaries:

Theorem 2.1 [3]. For a topological space $(X,\mathcal{T})$ and $A \subseteq X$, the following hold:

1- $Int(X - A) = X - Cl(A)$.
2- $Cl(X - A) = X - Int(A)$.

Theorem 2.2 [3]. Let A and B be two subset of a topological space $(X,\mathcal{T})$. If B is an open set in X then $Cl(A) \cap B \subseteq Cl(A \cap B)$.

Theorem 2.3 [7]. Every θ-closed set is closed set.

Definition 2.4 [2]. Anon-empty collection $\mathcal{G}$ of subsets of a topological space $(X,\mathcal{T})$ is said to be a grill on X if $\mathcal{G}$ satisfies following conditions:

1- $\phi \notin \mathcal{G}$.
2- $A \in \mathcal{G}$ and $A \subseteq B \Rightarrow B \in \mathcal{G}$.
3- $A, B \subseteq X$ and $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

For a topological space X, the operator φ: P(X) → P(X) from the power set P(X) of X to P(X) was first defined in [5], as $\varphi(A) = \{x \in X: U \cap A \in \mathcal{G}$, for each open set U containing x$\}$. The operator ψ(A): P(X) → P(X), is given by $\psi(A) = A \cup \varphi(A)$, for $A \in P(X)$. This operator was also in [6] to called a Kuratowski closure operator. So for a grill topological space $(X,\mathcal{T},\mathcal{G})$, there exists an unique topology $\mathcal{T}_{\mathcal{G}}$ on X. This topology defined by

$\mathcal{T}_{\mathcal{G}} = \{U \subseteq X : \psi(X - U) = X - U\}$.

For any $A \subseteq X$, $\psi(A) = {}_{\mathcal{G}}Cl(A)$ such that ${}_{\mathcal{G}}Cl(A)$ denotes the set of all $\mathcal{G}$-closure points of A in a topological space $(X, \mathcal{T}_{\mathcal{G}})$. The intersection of all closed subsets of $(X,\mathcal{T},\mathcal{G})$ containing A is denoted by ${}_{\mathcal{G}}Cl(A)$ and the interior set of A is defined as the union of all open subsets of $(X,\mathcal{T},\mathcal{G})$ contained in A and is denoted by ${}_{\mathcal{G}}Int(A)$.

* Faculty of Applied Sciences, Taiz University, Taiz, Yemen
** Faculty of Education , Aden University , Aden , Yemen.
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Theorem 2.5 [6]. Let $(X,\mathcal{T},\mathcal{G})$ be a grill topological space and $A,B \subseteq X$, the following properties hold:
1- $A \subseteq B$ implies that $\varphi(A) \subseteq \varphi(B)$.
2- $\varphi(A \cup B) = \varphi(A) \cup \varphi(B)$.
3- $\varphi(\varphi(A)) \subseteq \varphi(A) = Cl(\varphi(A)) \subseteq Cl(A)$.
4- if $U \in \mathcal{T}$ then $U \cap \varphi(A) \subseteq \varphi(U \cap A)$.
Theorem 2.6 [1]. If A is a subset of a grill topological space $(X,\mathcal{T},\mathcal{G})$ and U is an open set in $(X,\mathcal{T})$ then $U \cap \psi(A) \subseteq \psi(U \cap A)$.

$\mathcal{G}^S$-open sets:

Definition 3.1. A subset A of a grill topological space $(X,\mathcal{T},\mathcal{G})$ is said to be $\mathcal{G}^S$-open set if $A \subseteq Cl[{}_{\mathcal{G}}Int(\psi(A))]$. The complement of $\mathcal{G}^S$-open set is said to be $\mathcal{G}^S$-closed set.
For a grill topological space $(X,\mathcal{T},\mathcal{G})$, the set of all $\mathcal{G}^S$-open sets in X denoted by $\mathcal{G}^S O(X,\mathcal{T})$ and the set of all $\mathcal{G}^S$-closed sets in X denoted by $\mathcal{G}^S C(X,\mathcal{T})$.
Example 3.2. In a grill topological space $(X,\mathcal{T},\mathcal{G})$, where $X = \{a, b, c\}$, $\mathcal{T} = \{\phi, X, \{a, b\}\}$ and $\mathcal{G} = \{\{b\}, \{a, b\}, \{c, b\}, X\}$, $\mathcal{G}^S O(X,\mathcal{T}) = \{\phi, X, \{b\}, \{a, b\}, \{c, b\}\}$ and
$\mathcal{G}^S C(X,\mathcal{T}) = \{\phi, X, \{a, c\}, \{c\}, \{a\}\}$.
Theorem 3.3. A subset A of a grill topological space $(X,\mathcal{T},\mathcal{G})$ is $\mathcal{G}^S$-closed set if and only if $Int[\psi({}_{\mathcal{G}}Int(A))] \subseteq A$.
Proof. A is a $\mathcal{G}^S$-closed set in X if and only if $X - A$ is a $\mathcal{G}^S$-open set in X if and only if
$(X - A) \subseteq Cl[{}_{\mathcal{G}}Int(\psi(X - A))]$.
If and only if by using Theorem (2.1.),

$$\begin{aligned}(X - A) \subseteq Cl[{}_{\mathcal{G}}Int(\psi(X - A))] &= Cl[{}_{\mathcal{G}}Int({}_{\mathcal{G}}Cl(X - A))] \\ &= Cl[{}_{\mathcal{G}}Int(X - {}_{\mathcal{G}}Int(A))] = Cl[X - {}_{\mathcal{G}}Cl({}_{\mathcal{G}}Int(A))] \\ &= X - Int[{}_{\mathcal{G}}Cl({}_{\mathcal{G}}Int(A))] = X - Int[\psi({}_{\mathcal{G}}Int(A))].\end{aligned}$$

If and only if $Int[\psi({}_{\mathcal{G}}Int(A))] \subseteq A$.
Theorem 3.4. Let $(X,\mathcal{T},\mathcal{G})$ be a grill topological space. If A_λ is $\mathcal{G}^S$-open set for each $\lambda \in \Delta$ then $\cup_{\lambda\in\Delta} A_\lambda$ is $\mathcal{G}^S$-open set, where Δ is an index set.
Proof. Since A_λ is $\mathcal{G}^S$-open set for each $\lambda \in \Delta$, then $A_\lambda \subseteq Cl[{}_{\mathcal{G}}Int(\psi(A_\lambda))]$ for each $\lambda \in \Delta$. Then by Theorem (2.5.),

$$\begin{aligned}\cup_{\lambda\in\Delta} A_\lambda &\subseteq \cup_{\lambda\in\Delta} Cl[{}_{\mathcal{G}}Int(\psi(A_\lambda))] \subseteq Cl[\cup_{\lambda\in\Delta} {}_{\mathcal{G}}Int(\psi(A_\lambda))] \\ &\subseteq Cl[{}_{\mathcal{G}}Int(\cup_{\lambda\in\Delta} \psi(A_\lambda))] \subseteq Cl[{}_{\mathcal{G}}Int(\cup_{\lambda\in\Delta}(A_\lambda \cup \varphi(A_\lambda))] \\ &\subseteq Cl[{}_{\mathcal{G}}Int((\cup_{\lambda\in\Delta} A_\lambda) \cup (\cup_{\lambda\in\Delta} \varphi(A_\lambda))] \\ &\subseteq Cl[{}_{\mathcal{G}}Int(\cup_{\lambda\in\Delta} A_\lambda \cup \varphi(\cup_{\lambda\in\Delta} A_\lambda))] \\ &= Cl[{}_{\mathcal{G}}Int(\psi(\cup_{\lambda\in\Delta} A_\lambda))].\end{aligned}$$

Hence $\cup_{\lambda\in\Delta} A_\lambda$ is $\mathcal{G}^S$-open set.
Theorem 3.5. Let $(X,\mathcal{T},\mathcal{G})$ be a grill topological space. If U is an open set in $(X,\mathcal{T})$ and A is $\mathcal{G}^S$-open set then $U \cap A$ is $\mathcal{G}^S$-open set.
Proof. Since A is $\mathcal{G}^S$-open set then $A \subseteq Cl[{}_{\mathcal{G}}Int(\psi(A))]$. Then by Theorem (2.6.) and (2.2.),

$$\begin{aligned}U \cap A &\subseteq U \cap Cl[{}_{\mathcal{G}}Int(\psi(A))] \subseteq Cl[U \cap {}_{\mathcal{G}}Int(\psi(A))] \\ &= Cl[{}_{\mathcal{G}}Int(U) \cap {}_{\mathcal{G}}Int(\psi(A))] = Cl[{}_{\mathcal{G}}Int(U \cap \psi(A))] \\ &\subseteq Cl[{}_{\mathcal{G}}Int(\psi(U \cap A))].\end{aligned}$$

Hence $U \cap A$ is $\mathcal{G}^S$-open set.
For a grill topological space $(X,\mathcal{T},\mathcal{G})$ and a subset A of X, the $\mathcal{G}^S$–closure set of A is defined as the intersection of all $\mathcal{G}^S$-closed sets containing A and is denoted by ${}_{\mathcal{G}}{}^sCI(A)$. The $\mathcal{G}^S$-interior set of A is defined as the union of all $\mathcal{G}^S$-open sets of X contained in A and is denoted by ${}_{\mathcal{G}}{}^sInt(A)$. It is clear that ${}_{\mathcal{G}}{}^sCI(A)$ is a $\mathcal{G}^S$-closed subset of X and ${}_{\mathcal{G}}{}^sInt(A)$ is a $\mathcal{G}^S$-open subset of X.
For a subset $A \subseteq X$ of a grill topological space $(X,\mathcal{T},\mathcal{G})$, it is clear from the definition of ${}_{\mathcal{G}}{}^sCI(A)$ and ${}_{\mathcal{G}}{}^sInt(A)$ that $A \subseteq {}_{\mathcal{G}}{}^sCI(A)$ and ${}_{\mathcal{G}}{}^sInt(A) \subseteq A$.
Theorem 3.6. For a subset $A \subseteq X$ of grill topological space $(X,\mathcal{T},\mathcal{G})$, ${}_{\mathcal{G}}{}^sCI(A) = A$ if and only if A is a $\mathcal{G}^S$-closed set.
Proof. Let ${}_{\mathcal{G}}{}^sCI(A) = A$. Then from the definition of ${}_{\mathcal{G}}{}^sCI(A)$ and Theorem (3.3.), ${}_{\mathcal{G}}{}^sCI(A)$ is a $\mathcal{G}^S$-closed set and so A is a $\mathcal{G}^S$-closed set.
Conversely, we have $A \subseteq {}_{\mathcal{G}}{}^sCI(A)$. Since A is a $\mathcal{G}^S$-closed set, then it is clear from the definition of

${}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A)$, ${}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A) \subseteq A$. Hence $A = {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A)$.

Theorem 3.7. For a subset $A \subseteq X$ of grill topological space $(X, \mathcal{T}, \mathcal{G})$, ${}_{\mathcal{G}}{}^{s}\mathrm{Int}(A) = A$ if and only if A is a $\mathcal{G}^S$-open set.

Proof. Similar to the proof of Theorem (3.6.).

Theorem 3.8. For a subset $A \subseteq X$ of grill topological space $(X, \mathcal{T}, \mathcal{G})$, $x \in {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A)$ if and only if for all $\mathcal{G}^S$-open set U containing x, $U \cap A \neq \phi$.

Proof. Let $x \in {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A)$ and U be a $\mathcal{G}^S$-open set containing x. If $U \cap A = \phi$ then $A \subseteq X - U$. Since $X - U$ is a $\mathcal{G}^S$-closed set containing A, then ${}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A) \subseteq X - U$ and so $x \in {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A) \subseteq X - U$. Hence this is a contradiction, because $x \in U$. Therefore $U \cap A \neq \phi$.

Conversely, let $x \notin {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A)$. Then $X - {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A)$ is a $\mathcal{G}^S$-open set containing x. Hence by hypothesis, $[X - {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A)] \cap A \neq \phi$. But this is a contradiction, because $X - {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A) \subseteq X - A$.

Theorem 3.9. For a subset $A \subseteq X$ of grill topological space $(X, \mathcal{T}, \mathcal{G})$, $x \in {}_{\mathcal{G}}{}^{s}\mathrm{Int}(A)$ if and only if there is $\mathcal{G}^S$-open set U such that $x \in U \subseteq A$.

Proof. Let $x \in {}_{\mathcal{G}}{}^{s}\mathrm{Int}(A)$ and take $U = {}_{\mathcal{G}}{}^{s}\mathrm{Int}(A)$. Then by definition of ${}_{\mathcal{G}}{}^{s}\mathrm{Int}(A)$ we get that U is a $\mathcal{G}^S$-open set and $x \in U \subseteq A$.

Conversely, let there is $\mathcal{G}^S$-open set U such that $x \in U \subseteq A$. Then $x \in U \subseteq {}_{\mathcal{G}}{}^{s}\mathrm{Int}(A)$.

Theorem 3.10. For subsets $A,B \subseteq X$ of grill topological space $(X, \mathcal{T}, \mathcal{G})$, the following hold:

1- if $A \subseteq B$ then ${}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A) \subseteq {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(B)$.

2- ${}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A) \cup {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(B) \subseteq {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A \cup B)$.

3- ${}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A \cap B) \subseteq {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A) \cap {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(B)$.

4- ${}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A) \subseteq \mathrm{Cl}(A)$.

Proof. 1. Let $x \in {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A)$. Then by Theorem (3.8.), for all $\mathcal{G}^S$-open sets U containing x, $U \cap A \neq \phi$. Since $A \subseteq B$, then $U \cap B \neq \phi$. Hence $x \in {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(B)$. That is, ${}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A) \subseteq {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(B)$.

2. It is clear from the Part (1).

3. It is clear from the Part (1).

4. It is clear from Theorem (3.8.) and from every open set U is $\mathcal{G}^S$-open set.

Similar for the proof of the last theorem, we can proof the following theorem:

Theorem 3.11. For a subsets $A,B \subseteq X$ of grill topological space $(X, \mathcal{T}, \mathcal{G})$, the following hold:

1- if $A \subseteq B$ then ${}_{\mathcal{G}}{}^{s}\mathrm{Int}(A) \subseteq {}_{\mathcal{G}}{}^{s}\mathrm{Int}(B)$.

2- ${}_{\mathcal{G}}{}^{s}\mathrm{Int}(A) \cup {}_{\mathcal{G}}{}^{s}\mathrm{Int}(B) \subseteq {}_{\mathcal{G}}{}^{s}\mathrm{Int}(A \cup B)$.

3- ${}_{\mathcal{G}}{}^{s}\mathrm{Int}(A \cap B) \subset {}_{\mathcal{G}}{}^{s}\mathrm{Int}(A) \cap {}_{\mathcal{G}}{}^{s}\mathrm{Int}(B)$

4- $\mathrm{Int}(A) \subseteq {}_{\mathcal{G}}{}^{s}\mathrm{Int}(A)$.

Theorem 3.12. For a subset $A \subseteq X$ of grill topological space $(X, \mathcal{T}, \mathcal{G})$, the following hold:

1- ${}_{\mathcal{G}}{}^{s}\mathrm{Int}(X - A) = X - {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A)$.

2- ${}_{\mathcal{G}}{}^{s}\mathrm{Cl}(X - A) = X - {}_{\mathcal{G}}{}^{s}\mathrm{Int}(A)$.

Proof. 1. Since $A \subseteq {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A)$ then $X - {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A) \subseteq X - A$. Since $X - {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A)$ is a $\mathcal{G}^s$-open set in $(X, \mathcal{T}, \mathcal{G})$ then $X - {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A) = {}_{\mathcal{G}}{}^{s}\mathrm{Int}[X - {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A)] \subseteq {}_{\mathcal{G}}{}^{s}\mathrm{Int}(X - A)$.

For the other side, let $x \in {}_{\mathcal{G}}{}^{s}\mathrm{Int}(X - A)$. Then there is $\mathcal{G}^S$-open set U such that $x \in U \subseteq X - A$. Then $X - U$ is a $\mathcal{G}^S$-closed set containing A and $x \notin X - U$. Hence $x \notin {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A)$, that is, $x \in X - {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A)$.

7. Since ${}_{\mathcal{G}}{}^{s}\mathrm{Int}(A) \subseteq A$ then $X - A \subseteq X - {}_{\mathcal{G}}{}^{s}\mathrm{Int}(A)$. Since $X - {}_{\mathcal{G}}{}^{s}\mathrm{Int}(A)$ is a $\mathcal{G}^S$-closed set in $(X, \mathcal{T}, \mathcal{G})$, then ${}_{\mathcal{G}}{}^{s}\mathrm{Cl}(X - A) \subseteq {}_{\mathcal{G}}{}^{s}\mathrm{Cl}[X - {}_{\mathcal{G}}{}^{s}\mathrm{Int}(A)] = X - {}_{\mathcal{G}}{}^{s}\mathrm{Int}(A)$.

For the other side, let $x \in {}_{\mathcal{G}}{}^{s}\mathrm{Int}(X - A)$. Then there is $\mathcal{G}^S$-open set U such that $x \in U \subseteq X - A$. Then $X - U$ is a $\mathcal{G}^S$-closed set containing A and $x \notin X - U$. Hence $x \notin {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A)$, that is, $x \in X - {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A)$.

Theorem 3.13. For a subset $A \subseteq X$ of grill topological space $(X, \mathcal{T}, \mathcal{G})$, the following hold:

1- if G is an open set of X then ${}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A) \cap G \subseteq {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A \cap G)$.

2- if G is a closed set of X then ${}_{\mathcal{G}}{}^{s}\mathrm{Int}(A \cup G) \subseteq {}_{\mathcal{G}}{}^{s}\mathrm{Int}(A) \cup G$.

Proof. 1. Let $x \in {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A) \cap G$. Then $x \in {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A)$ and $x \in G$. Let V be any $\mathcal{G}^S$-open set in $(X, \mathcal{T}, \mathcal{G})$ containing x. By Theorem (2.1.11), $V \cap G$ is $\mathcal{G}^S$-open set containing x. Since $x \in {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A)$, then by Theorem (3.8.), $(V \cap G) \cap A \neq \phi$. This implies, $V \cap (G \cap A) \neq \phi$. Hence by Theorem (3.8.), $x \in {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A \cap G)$. That is, ${}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A) \cap G \subseteq {}_{\mathcal{G}}{}^{s}\mathrm{Cl}(A \cap G)$.

7. Since G is a closed set of X then by part (1) and Theorem (3.12.),

$$\begin{aligned} X - [{}_{\mathcal{G}}^{s}\mathrm{Int}(A) \cup G] = [X - {}_{\mathcal{G}}^{s}\mathrm{Int}(A)] \cap [X - G] &= [{}_{\mathcal{G}}^{s}\mathrm{Cl}(X - A)] \cap [X - G] \\ &\subseteq {}_{\mathcal{G}}^{s}\mathrm{Cl}[(X - A) \cap (X - G)] \\ &\subseteq {}_{\mathcal{G}}^{s}\mathrm{Cl}(X - A) \cap {}_{\mathcal{G}}^{s}\mathrm{Cl}(X - G) \\ &= {}_{\mathcal{G}}^{s}\mathrm{Cl}(X - A) \cap (X - G) \\ &= (X - {}_{\mathcal{G}}^{s}\mathrm{Int}(A)) \cap (X - G) \\ &= X - ({}_{\mathcal{G}}^{s}\mathrm{Int}(A) \cup G). \end{aligned}$$

Hence ${}_{\mathcal{G}}^{s}\mathrm{Int}(A \cup G) \subseteq {}_{\mathcal{G}}^{s}\mathrm{Int}(A) \cup G$.

4. $\boldsymbol{\theta}$-$\boldsymbol{\mathcal{G}^S}$-closed set:

Let $(X, \mathcal{T}, \mathcal{G})$ be a grill topological space and $A \subseteq X$. A point $x \in X$ is called θ-$\mathcal{G}^S$-cluster point of A if ${}_{\mathcal{G}}^{s}\mathrm{Cl}(U) \cap A \neq \phi$, for every $\mathcal{G}^S$-open set U in $(X, \mathcal{T}, \mathcal{G})$ containing x. The set of all θ-$\mathcal{G}^S$-cluster points of A is called the θ-$\mathcal{G}^S$-cluster set of A and denoted by ${}_{\mathcal{G}}^{s}\mathrm{Cl}^{\theta}(A)$.

Definition 4.2. A subset A of grill topological space $(X, \mathcal{T}, \mathcal{G})$ is called θ-$\mathcal{G}^S$–closed set in $(X, \mathcal{T}, \mathcal{G})$ if ${}_{\mathcal{G}}^{s}\mathrm{Cl}^{\theta}(A) = A$. The complement of θ-$\mathcal{G}^S$-closed set in $(X, \mathcal{T}, \mathcal{G})$ is called θ-$\mathcal{G}^S$-open set in $(X, \mathcal{T}, \mathcal{G})$.

Theorem 4.3. Every θ-closed set in a space $(X, \mathcal{T})$ is θ-$\mathcal{G}^S$-closed set in grill topological space $(X, \mathcal{T}, \mathcal{G})$.

Proof. Let A be a θ-closed set in a space $(X, \mathcal{T})$, that is, $\mathrm{Cl}^{\theta}(A) = A$. It is clear that $A \subseteq {}_{\mathcal{G}}^{s}\mathrm{Cl}^{\theta}(A)$. We prove that ${}_{\mathcal{G}}^{s}\mathrm{Cl}^{\theta}(A) \subseteq A$. Let $x \in {}_{\mathcal{G}}^{s}\mathrm{Cl}^{\theta}(A)$. Then $U \cap A \neq \phi$, for every $\mathcal{G}^S$-open set U of $(X, \mathcal{T}, \mathcal{G})$ containing x, since $U \subseteq {}_{\mathcal{G}}^{s}\mathrm{Cl}(U)$, then ${}_{\mathcal{G}}^{s}\mathrm{Cl}(U) \cap A \neq \phi$, for every $\mathcal{G}^S$-open set U in $(X, \mathcal{T}, \mathcal{G})$ containing x. Since ${}_{\mathcal{G}}^{s}\mathrm{Cl}(U) \subseteq \mathrm{Cl}(U)$ then $\mathrm{Cl}(U) \cap A \neq \phi$, for every $\mathcal{G}^S$-open set U in $(X, \mathcal{T}, \mathcal{G})$ containing x. Then $x \in \mathrm{Cl}^{\theta}(A) = A$. Hence ${}_{\mathcal{G}}^{s}\mathrm{Cl}^{\theta}(A) \subseteq A$. That is, A is a θ-$\mathcal{G}^S$-closed set in grill topological space $(X, \mathcal{T}, \mathcal{G})$.

The converse of the last theorem need not be true.

Example 4.4. In a grill topological space $(X, \mathcal{T}, \mathcal{G})$, where X= {a,b,c}, $\mathcal{T} = \{\phi, X, \{a, b\}\}$ and $\mathcal{G} = \{\{c\}, \{a, c\}, \{b, c\}, X\}$, the set {a} is a θ-$\mathcal{G}^S$-closed set in $(X, \mathcal{T}, \mathcal{G})$ but it is not θ-closed set in $(X, \mathcal{T})$.

Theorem 4.5. Every θ-$\mathcal{G}^S$-closed set is $\mathcal{G}^S$-closed set.

Proof. Let $(X, \mathcal{T}, \mathcal{G})$ be a grill topological space and A be a θ-$\mathcal{G}^S$-closed set, that is, ${}_{\mathcal{G}}^{s}\mathrm{Cl}^{\theta}(A) = A$. It is clear that $A \subseteq {}_{\mathcal{G}}^{s}\mathrm{Cl}(A)$. We prove that ${}_{\mathcal{G}}^{s}\mathrm{Cl}(A) \subseteq A$. Let $x \in {}_{\mathcal{G}}^{s}\mathrm{Cl}(A)$. Then $U \cap A \neq \phi$, for every $\mathcal{G}^S$-open set U in $(X, \mathcal{T}, \mathcal{G})$ containing x. Since $U \subseteq {}_{\mathcal{G}}^{s}\mathrm{Cl}(U)$ then ${}_{\mathcal{G}}^{s}\mathrm{Cl}(U) \cap A \neq \phi$, for every $\mathcal{G}^S$-open set U in $(X, \mathcal{T}, \mathcal{G})$ containing x. Then $x \in {}_{\mathcal{G}}^{s}\mathrm{Cl}^{\theta}(A) = A$. Hence ${}_{\mathcal{G}}^{s}\mathrm{Cl}(A) \subseteq A$. That is, A is a $\mathcal{G}^S$-closed set in grill topological space $(X, \mathcal{T}, \mathcal{G})$.

The converse of Theorem 4.5. need not be true.

Example 4.6. Let $(X, \mathcal{T}, \mathcal{G})$ be a grill topological space. X= {a, b, c}, $\mathcal{T} = \{\phi, X, \{a, b\}\}$ and $\mathcal{G} = P(X) - \{\phi\}$. The set {b} is a $\mathcal{G}^S$-closed set in $(X, \mathcal{T}, \mathcal{G})$ but it is not θ-$\mathcal{G}^S$-closed set, where P(X) is the power set of X.

Theorem 4.7. For every $\mathcal{G}^S$-open set G in grill topological space $(X, \mathcal{T}, \mathcal{G})$, ${}_{\mathcal{G}}^{s}\mathrm{Cl}^{\theta}(G) = {}_{\mathcal{G}}^{s}\mathrm{Cl}(G)$.

Proof. Let $x \in {}_{\mathcal{G}}^{s}\mathrm{Cl}(G)$. Then for every $\mathcal{G}^S$-open set U in $(X, \mathcal{T}, \mathcal{G})$ containing x, $U \cap G \neq \phi$. Since $U \subseteq {}_{\mathcal{G}}^{s}\mathrm{Cl}(U)$ then ${}_{\mathcal{G}}^{s}\mathrm{Cl}(U) \cap G \neq \phi$. Hence $x \in {}_{\mathcal{G}}^{s}\mathrm{Cl}^{\theta}(U)$. That is, ${}_{\mathcal{G}}^{s}\mathrm{Cl}(G) \subseteq {}_{\mathcal{G}}^{s}\mathrm{Cl}^{\theta}(G)$. For the other side, let $x \in {}_{\mathcal{G}}^{s}\mathrm{Cl}^{\theta}(G)$. Then for every $\mathcal{G}^S$-open set U in $(X, \mathcal{T}, \mathcal{G})$ containing x, ${}_{\mathcal{G}}^{s}\mathrm{Cl}(U) \cap G \neq \phi$.

Since G is $\mathcal{G}^S$-open set U in $(X, \mathcal{T}, \mathcal{G})$. Then by Theorem (3.13.), ${}_{\mathcal{G}}^{s}\mathrm{Cl}(U) \cap G = {}_{\mathcal{G}}^{s}\mathrm{Cl}(U \cap G)$. Then ${}_{\mathcal{G}}^{s}\mathrm{Cl}(U \cap G) \neq \phi$. Hence $U \cap G \neq \phi$. That is, $x \in {}_{\mathcal{G}}^{s}\mathrm{Cl}(U)$. Hence ${}_{\mathcal{G}}^{s}\mathrm{Cl}^{\theta}(G) \subseteq {}_{\mathcal{G}}^{s}\mathrm{Cl}(G)$.

From Theorems (4.3) and (4.5) we have the following between θ-$\mathcal{G}^S$-closed sets and some other known sets.

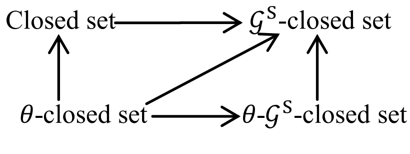

diagram 1

Theorem 4.8. A subset U is θ-$\mathcal{G}^S$-open set in grill topological space $(X, \mathcal{T}, \mathcal{G})$ if and only if for each $x \in U$ there is $\mathcal{G}^S$-open set V in $(X, \mathcal{T}, \mathcal{G})$ containing x such that ${}_{\mathcal{G}}^{s}\mathrm{Cl}(V) \subseteq U$.

Proof. Suppose that U is θ-$\mathcal{G}^S$-open set in $(X, \mathcal{T}, \mathcal{G})$ and $x \in U$. Then $x \notin X - U = {}_{\mathcal{G}}^{s}\mathrm{Cl}^{\theta}(X - U)$. Then there is $\mathcal{G}^S$-open set V in $(X, \mathcal{T}, \mathcal{G})$ containing x such that ${}_{\mathcal{G}}^{s}Cl(V) \cap (X - U) = \phi$. That is, ${}_{\mathcal{G}}^{s}Cl(V) \subseteq U$.

Conversely, suppose that U is not θ-$\mathcal{G}^S$-open set. Then X – U is not θ-$\mathcal{G}^S$–closed set. That is, there is $x \in$

${}_{\mathcal{G}}{}^{s}Cl^{\theta}$(X − U) and x ∉ X − U. Since x ∈ U then by the hypothesis, there is $\mathcal{G}^S$-open set V in $(X,\mathcal{T},\mathcal{G})$ containing x such that ${}_{\mathcal{G}}{}^{s}CI$(V) ⊆ U. This implies, ${}_{\mathcal{G}}{}^{s}CI$(V) ∩ (X−U) = ϕ and this contradiction since x ∈ ${}_{\mathcal{G}}{}^{s}Cl^{\theta}$(X−U). Hence U is θ- $\mathcal{G}^S$-open set.

θ-$\mathcal{G}^S$-continuous function:

Definition 5.1. A function f☹$X,\mathcal{T},\mathcal{G}) \to (Y,P)$ of a grill topological space $(X,\mathcal{T},\mathcal{G})$ into a space (Y,P) is called θ-$\mathcal{G}^S$-continuous function if for each x ∈ X and each open set V in (Y,P) containing f(x), there exists $\mathcal{G}^S$-open set U in $(X,\mathcal{T},\mathcal{G})$ containing x such that f(${}_{\mathcal{G}}{}^{s}CI$(U)) ⊆ ${}_{P}Cl$(V).

Theorem 5.2. A function f☹$X,\mathcal{T},\mathcal{G}) \to (Y,P)$ is θ-$\mathcal{G}^S$-continuous if and only if ${}_{\mathcal{G}}{}^{s}Cl^{\theta}(f^{-1}$(V)) ⊆ f^{-1}(${}_{P}Cl$(V)), for every open set V in (Y,P).

Proof. Suppose that f is θ-$\mathcal{G}^S$-continuous. Let V be any open set in (Y,P). Let x ∉ f^{-1}(${}_{P}Cl$(V)). Then f(x) ∉ ${}_{P}Cl$(V). Then f(x) ∈ Y−${}_{P}Cl$(V). Since Y− ${}_{P}Cl$(V) is open set in (Y,P) containing x and f is θ-$\mathcal{G}^S$-continuous then there exists $\mathcal{G}^S$-open set U in $(X,\mathcal{T},\mathcal{G})$ containing x such that f(${}_{\mathcal{G}}{}^{s}CI$(U)) ⊆ ${}_{P}Cl$(Y − ${}_{P}Cl$(V)). This implies,

f(${}_{\mathcal{G}}{}^{s}CI$(U)) ⊆ ${}_{P}Cl$(Y − ${}_{P}Cl$(V)) = Y − ${}_{p}Int$(${}_{P}Cl$(V)).

Hence f(${}_{\mathcal{G}}{}^{s}CI$(U)) ∩ ${}_{p}Int$(${}_{P}Cl$(V)) = ϕ. Since V = ${}_{p}Int$(V) ⊆ ${}_{p}Int$(${}_{P}Cl$(V)) then f(${}_{\mathcal{G}}{}^{s}CI$(U)) ∩ V = ϕ and so ${}_{\mathcal{G}}{}^{s}CI$(U) ∩ f^{-1}(V) = ϕ. Since U is $\mathcal{G}^S$-open set in $(X,\mathcal{T},\mathcal{G})$ containing x then x ∉ ${}_{\mathcal{G}}{}^{s}Cl^{\theta}(f^{-1}$(V)). Hence ${}_{\mathcal{G}}{}^{s}Cl^{\theta}(f^{-1}$(V)) ⊆ f^{-1}(${}_{P}Cl$(V)).

Conversely, Let x ∈ X be any point in X and V be any open set of (Y,P) containing f(x). Since V ∩ (Y−${}_{P}Cl$(V)) = ϕ then f(x) ∉ ${}_{P}Cl$(Y−${}_{P}Cl$(V)). This implies, x ∉ f^{-1}[${}_{P}Cl$(Y −${}_{P}Cl$(V))]. Since Y − ${}_{P}Cl$(V) is an open set in (Y,P) then by the hypothesis,

${}_{\mathcal{G}}{}^{s}Cl^{\theta}$[$f^{-1}$(Y − ${}_{P}Cl$(V))] ⊆ f^{-1}[${}_{P}Cl$(Y − ${}_{P}Cl$(V))].

Then x ∉ ${}_{\mathcal{G}}{}^{s}Cl^{\theta}$[$f^{-1}$(Y − ${}_{P}Cl$(V))]. Hence there is $\mathcal{G}^S$-open set U in $(X,\mathcal{T},\mathcal{G})$ containing x such that ${}_{\mathcal{G}}{}^{s}CI$(U) ∩ f^{-1}(Y − ${}_{P}Cl$(V)) = ϕ. This implies, f(${}_{\mathcal{G}}{}^{s}CI$(U)) ⊆ ${}_{P}Cl$(V). Hence f is θ-$\mathcal{G}^S$-continuous.

Theorem 5.3. A function f☹$X,\mathcal{T},\mathcal{G}) \to (Y,P)$ is θ-$\mathcal{G}^S$-continuous if and only if

${}_{\mathcal{G}}{}^{s}Cl^{\theta}$[X − f^{-1}(${}_{P}Cl$(V))] ⊆ X − f^{-1}(V)

for every open set V in (Y,P).

Proof. Suppose that f is θ-$\mathcal{G}^S$-continuous. Let V be any open set in (Y,P). Let x ∉ X−f^{-1}(V). Then f(x) ∈ V. Since f is θ-$\mathcal{G}^S$-continuous then there exists $\mathcal{G}^S$-open set U in $(X,\mathcal{T},\mathcal{G})$ containing x such that f(${}_{\mathcal{G}}{}^{s}CI$(U)) ⊆ ${}_{P}Cl$(V). This implies, ${}_{\mathcal{G}}{}^{s}CI$(U) ⊆ f^{-1}(${}_{P}Cl$(V)). Then

${}_{\mathcal{G}}{}^{s}CI$(U) ∩ [X − f^{-1}(${}_{P}Cl$(V))] = ϕ.

Since U is a $\mathcal{G}^S$-open set in $(X,\mathcal{T},\mathcal{G})$ containing x then x ∉ ${}_{\mathcal{G}}{}^{s}Cl^{\theta}$[X−$f^{-1}$(${}_{P}Cl$(V))]. Hence

${}_{\mathcal{G}}{}^{s}Cl^{\theta}$[X − f^{-1}(${}_{P}Cl$(V))] ⊆ X − f^{-1}(V).

Conversely, let x ∈ X be any point in X and V be any open set in (Y,P) containing f(x). Then x ∈ f^{-1}(V), that is, x ∉ X−f^{-1}(V). Then by the hypothesis, x ∉ ${}_{\mathcal{G}}{}^{s}Cl^{\theta}$[X−$f^{-1}$(${}_{P}Cl$(V))]. That is, $\mathcal{G}^S$-open set U in $(X,\mathcal{T},\mathcal{G})$ containing x such that

${}_{\mathcal{G}}{}^{s}CI$(U) ∩ [X − f^{-1}(${}_{P}Cl$(V))] = ϕ.

This implies, ${}_{\mathcal{G}}{}^{s}CI$(U) ⊆ f^{-1}(${}_{P}Cl$(V)) and so f(${}_{\mathcal{G}}{}^{s}CI$(U)) ⊆ ${}_{P}Cl$(V). Hence f is θ-$\mathcal{G}^S$-continuous.

Theorem 5.4. For a function f☹$X,\mathcal{T},\mathcal{G}) \to (Y,P)$, the following conditions are equivalent:

1. f is θ-$\mathcal{G}^S$-continuous.
2. ${}_{\mathcal{G}}{}^{s}Cl^{\theta}(f^{-1}$(B)) ⊆ f^{-1}(${}_{\mathcal{G}}{}^{s}Cl^{\theta}$(B)), for every subset B ⊆ Y.
3. f(${}_{\mathcal{G}}{}^{s}Cl^{\theta}$(A)) ⊆ ${}_{p}Cl^{\theta}(f$(A)), for every subset A ⊆ X.

Proof. (1) ⟹ (2): Let B be any subset of Y. Suppose that x ∉ f^{-1}(${}_{p}Cl^{\theta}$(B)). Then$f$(x) ∉ ${}_{p}Cl^{\theta}$(B). Then there is an open set V in Y containing f(x) such that ${}_{P}Cl$(V) ∩ B =ϕ. Since f is θ-$\mathcal{G}^S$-continuous then there exists $\mathcal{G}^S$-open set U in $(X,\mathcal{T},\mathcal{G})$ containing x such that f(${}_{\mathcal{G}}{}^{s}CI$(U)) ⊆ ${}_{P}Cl$(V). Then we have f(${}_{\mathcal{G}}{}^{s}CI$(U)) ∩ B = ϕ. This implies, ${}_{\mathcal{G}}{}^{s}CI$(U) ∩ f^{-1}(B) = ϕ. Hence x ∉ ${}_{\mathcal{G}}{}^{s}Cl^{\theta}(f^{-1}$(B)). That is,

${}_{\mathcal{G}}{}^{s}Cl^{\theta}(f^{-1}$(B)) ⊆ f^{-1}(${}_{p}Cl^{\theta}$(B)).

(2) ⟹ (1): Let x ∈ X be any point in X and V be any open set in (Y,P) containing f(x). Since ${}_{P}Cl$(V) ∩ (Y−${}_{P}Cl$(V)) = ϕ then f(x) ∉ ${}_{p}Cl^{\theta}$(Y−${}_{P}$Cl(V)). This implies, x ∉ f^{-1}[${}_{p}Cl^{\theta}$(Y− ${}_{P}$Cl(V))]. Since ${}_{p}Cl^{\theta}$(Y − ${}_{P}$Cl(V)) ⊆ Y then by the hypothesis,

${}_{\mathcal{G}}{}^{s}Cl^{\theta}$[$f^{-1}$(${}_{p}Cl^{\theta}$(Y − ${}_{P}$Cl(V)))] ⊆ f^{-1}[${}_{p}Cl^{\theta}$(${}_{p}Cl^{\theta}$(Y − ${}_{P}$Cl(V)))]

= f^{-1}[${}_{p}Cl^{\theta}$(Y − ${}_{P}$Cl(V))].

Then x $\notin {}_{\mathcal{G}^s}\mathrm{Cl}^{\theta}[f^{-1}({}_{p}\mathrm{Cl}^{\theta}(Y - {}_{P}\mathrm{Cl}(V)))]$. Hence there is $\mathcal{G}^{S}$-open set U in $(X, \mathcal{T}, \mathcal{G})$ containing x such that ${}_{\mathcal{G}^s}\mathrm{CI}(U) \cap f^{-1}[{}_{p}\mathrm{Cl}^{\theta}(Y - {}_{P}\mathrm{Cl}(V))] = \phi$. This implies, $f({}_{\mathcal{G}^s}\mathrm{CI}(U)) \subseteq {}_{P}\mathrm{Cl}(V)$. Hence f is θ-$\mathcal{G}^{S}$-continuous.

(2) $\Longrightarrow$ (3): Let A be any subset of X. Since $f(A) \subseteq Y$ then by the hypothesis,

${}_{\mathcal{G}^s}\mathrm{Cl}^{\theta}(A) \subseteq {}_{\mathcal{G}^s}\mathrm{Cl}^{\theta}[f^{-1}(f(A))] \subseteq f^{-1}[{}_{p}\mathrm{Cl}^{\theta}(f(A))]$.

This implies, $f({}_{\mathcal{G}^s}\mathrm{Cl}^{\theta}(A)) \subseteq {}_{p}\mathrm{Cl}^{\theta}(f(A))$.

(3) $\Longrightarrow$ (2): Let B be any subset of Y. Since $f^{-1}(B) \subseteq X$ then by the hypothesis,

$f[{}_{\mathcal{G}^s}\mathrm{Cl}^{\theta}(f^{-1}(B))] \subseteq {}_{p}\mathrm{Cl}^{\theta}[f(f^{-1}(B))] \subseteq {}_{p}\mathrm{Cl}^{\theta}(B)$.

This implies, ${}_{\mathcal{G}^s}\mathrm{Cl}^{\theta}(f^{-1}(B)) \subseteq f^{-1}({}_{p}\mathrm{Cl}^{\theta}(B))$.

References:

1- Al-Omari , A . and Noiri , T ., (2011) , Decompositions of continuity Via grills, Jord. J. Math. Stat., 4 (33 – 46).
2- Choquet, G., (1947), Sur les notions de filter grille, competes Rend us Acad. Sci. Paris., 224 (171 – 173).
3- Dugundji, J., (1966), Topology. New Jersey: Allyn and Bacon.
4- Fomin, S., (1943), Extensions of topological space, Ann. Of Math., 44 (471 – 480).
5- Levine, N., (1961), A decomposition of continuity in topological spaces, Amer Math. Monthly., 68 (44 – 46).
6- Roy, B., and Mukherjee, M.N., (2007a), On a typical topology induced by a grill, Soochow J. Math., 33(4) (771 – 786.
7- Velicko, N., (1968), h-closed topological spaces, Amer. Math. Soc. Transl. Ser., 78 (103 – 118).

حول المجموعات- $\theta-\mathcal{G}^{s}$ – المغلقة والدوال $\theta-\mathcal{G}^{s}$ - المستمرة

أمين أحمد سيف ناهد محمد عبده أحمد

الملخص

الهدف من هذا البحث هو تقديم ودراسة صفوف جديدة في الفضاءات التبولوجية المشوية للمجموعات المغلقة θ والدوال المستمرة θ المدروسة في الفضاءات التبولوجية العادية من قبل الباحثين.

الكلمات الدليلية: نقاط الانغلاق θ . المجموعات المغلقة θ . الدوال المستمرة θ