q-Analogue Modified Laguerre and Generalized Laguerre Polynomials of Two Variables

Fadhle Bin Fadhle Mohsen* **Fadhl Saleh Nasser Alsarahi****

Abstract

The q-Laguerre polynomials are important q-orthogonal polynomials whose applications and generalizations arise in many applications such as quantumgroup (oscillator algebra, etc.), q-harmonic oscillator and coding theory.

In this paper, we introduce the q-analogue modified Laguerre and generalized modified Laguerre polynomials of two variables . Some recurrence relations for these polynomials are derived.

Keywords: q-analogue generalized modified Laguerre polynomials, generating functions and recurrence relations.

Introduction, definitions and notations:

In this section, we will give a summary of the mathematical notations and definitions required in this paper for the convenience of the reader.

The basic hypergeometric or q-hypergeometric function ${}_r\phi_s$ is defined by the series [3]

$$ {}_r\phi_s\left[\begin{matrix} a_1,\ldots,a_r; & \\ & q,z \\ b_1,\ldots,b_s; & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1,\ldots,a_r,q)_n}{(b_1,\ldots,b_s,q)_n} (-1)^{(1+s-r)n} q^{(1+s-r)\binom{n}{2}} \frac{z^n}{(q,q)_n}, \tag{1.1} $$

where $(a_1,\ldots,a_r,q)_n = (a_1;q)_n \ldots (a_r;q)_n$.

The q-analogues of Pochhammer symbol or q-shifted factorial be defined by [3]

$$ (a;q)_n = \begin{cases} 1 & , n=0 \\ \prod\limits_{0\le j\le n-1} (1-aq^j) & , n=1,2,3,\ldots \end{cases}, \tag{1.2} $$

where

$$ (q^{-n};q)_k = \begin{cases} 0 & k>n \\ \dfrac{(q;q)_n}{(q;q)_{n-k}} (-1)^k q^{\binom{k}{2}-nk}, & k\le n \end{cases}, \tag{1.3} $$

$$ (0;q)_n = 1, $$

also

$$ (a;q)_{n+k} = (a;q)_n (aq^n;q)_k, \tag{1.4} $$

where $\lim\limits_{q\to 1^-} \dfrac{(q^z;q)_k}{(1-q)^k} = (z)_k$.

* Department of Mathematics Faculty of education – Zingbar University of Aden Yafea.

** Department of Mathematics-Physice Faculty of education – University of Aden Yafea...Received on 20/9/2017 and Accepted for Publication on 8/10/2018

The q-binomial coefficient is defined by [3]

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \ , \quad 0 \le k \le n, \quad k,n \in N \ ; \qquad (1.5)$$

$$\begin{bmatrix} -n \\ k \end{bmatrix}_q = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(-q^{-n}\right)^k q^{-\binom{k}{2}}, \quad n \in C; k \in N_0, \qquad (1.6)$$

where C complex plane and $N_0 = \{0,1,2,3,\ldots\}$.

The q-exponential function $e_q(x)$ is defined by [3]

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}, \qquad e_1(x) = e\,(x), \qquad (1.7)$$

and

$$E_q(x) = \sum_{n\ge 0} (-1)^n q^{\binom{n}{2}} \frac{x^n}{(q;q)_n} = (x;q)_\infty . \qquad (1.8)$$

The q-derivative with index α is defined by [8]

$$D_\alpha f(x) = \frac{f(q^\alpha x) - f(x)}{(q^\alpha - 1)x}, \qquad D_1 = D \qquad (1.9)$$

which for q-derivative of the pair of functions are valid:

$$D(\lambda a(x) + \mu b(x)) = \lambda Da(x) + \mu Db(x), \qquad (1.10)$$

$$D(a(x).b(x)) = a(qx) D b(x) + D a(x) b(x), \qquad (1.11)$$

$$D\left(\frac{a(x)}{b(x)}\right) = \frac{Da(x)b(x) - a(x)Db(x)}{b(x)b(qx)}. \qquad (1.12)$$

Exton [2] presented the following q-exponential functions:

$$E(\mu, z; q) = \sum_{n=0}^{\infty} \frac{q^{\mu n(n-1)}}{[n]_q!} z^n ,$$

where $\quad [n]_q! = \dfrac{(q;q)_n}{(1-q)^n}$.

In Exton's formula, if we replace z by $\dfrac{x}{1-q}$ and μ by $2a$, we get

$$E\left(2a, \frac{x}{1-q}; q\right) = E_q(x,a),$$

where

$$E_q(x,a) = \sum_{n=0}^{\infty} \frac{q^{a\binom{n}{2}}}{(q;q)_n} x^n , \qquad (1.13)$$

which satisfies the functional relation

$$E_q(x,a) - E_q(qx,a) = xE_q(q^a x, a).$$

The above q-function can be rewritten by the formula

$$D_q E_q(x,a) = \frac{1}{1-q} E_q\left(q^a x, a\right). \tag{1.14}$$

Also, the q-analogue of power (binomial) function $(x \pm y)^n$ is given by [6]

$$(x \pm y)^n = (x \pm y)_n = x^n \left(\mp y/x ; q \right)_n = x^n \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} \left(\pm y/x \right)^k . \tag{1.15}$$

The Laguerre polynomials $L_n(x)$ of n order are defined by means of a generating relation [7]

$$(1-t)^{-1} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(x) t^n, \quad |t| < 1, \ 0 < x < \infty \tag{1.16}$$

and the following series definition

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n! x^k}{(k!)^2 (n-k)!}. \tag{1.17}$$

Also, the associated Laguerre polynomials are defined by the generating function [7]

$$(1-t)^{-1-\alpha} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n, \tag{1.18}$$

and the series definition

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n x^k}{k!(n-k)!(1+\alpha)_k}. \tag{1.19}$$

The two variable Laguerre polynomials are defined by the generating function [1]:

$$(1-yt)^{-1} \exp\left(\frac{-xt}{1-yt}\right) = \sum_{n=0}^{\infty} L_n(x,y) t^n, \tag{1.20}$$

or equivalcutly

$$(1-yt)^{-1-\alpha} \exp\left(\frac{-xt}{1-yt}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x,y) t^n \quad . \tag{1.21}$$

The two variables Laguerre polynomials are defined by the series definition

$$L_n(x,y) = n! \sum_{k=0}^{n} \frac{(-1)^k x^k y^{n-k}}{(k!)^2 (n-k)!}, \tag{1.22}$$

Further, the two variable associated Laguerre polynomials are defined by [1]

$$L_n^{(\alpha)}(x,y) = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n x^k y^{n-k}}{k!(n-k)!(1+\alpha)_k}. \tag{1.23}$$

Khan [4] is defined the generating relation for the two variable modified Laguerre polynomials by follows

$$\sum_{n=0}^{\infty} L_{\alpha,\beta,m,n}(x,y) t^n = (1-\beta t y)^{-m} \exp\left(\frac{-\alpha x t}{1-\beta y}\right), \tag{1.24}$$

where two variable modified Laguerre polynomials $L_{\alpha,\beta,m,n}(x,y)$ is given by

$$L_{\alpha,\beta,m,n}(x,y) = \frac{(m)_n (\beta y)^n}{n!} {}_1F_1\left(-n; m; \frac{\alpha x}{\beta y}\right). \tag{1.25}$$

The q-Laguerre polynomials are defined by [5]

$$L_n^{(\alpha)}(x;q)=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}{}_1\phi_1\begin{pmatrix} q^{-n}; & \\ & q;-(1-q)q^{\alpha+n+1}x \\ q^{\alpha+1}; & \end{pmatrix}$$

$$=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^{\infty}\frac{(q^{-n};q)_k q^{\binom{k}{2}}(1-q)^k (q^{n+\alpha+1}x)^k}{(q^{\alpha+1};q)_k (q;q)_k} \tag{1.26}$$

where $\alpha>-1,\quad 0<q<1$ and $n=0,1,2,3,\ldots$.

The q-Laguerre polynomials are specified by the following generating function:

$$\sum_{n=0}^{\infty}L_n^{(\alpha)}(x;q)t^n=\frac{1}{(t;q)_\infty}{}_1\phi_1\left(\begin{matrix}-x\\0\end{matrix}\Big/ q;q^{\alpha+1}t\right). \tag{1.27}$$

1. q-Analogue Modified Laguerre Polynomials of Two Variables

We introduce q-analogue modified Laguerre polynomial of two variables by the following:

$$L_{m,n}^{(\alpha,\beta)}(x,y;q)=\frac{(q^m;q)_n(\beta y)^n}{q^{mn}(q;q)_n}{}_1\phi_1\left(q^{-n};q^m;q,-q^{m+1}\frac{\alpha x}{\beta y}\right). \tag{2.1}$$

Now, we get generating function of the q-analogue modified Laguerre polynomials in the form of the following theorem:

Theorem 2.1

The following generating function for the q-analogue Laguerre polynomials $L_{m,n}^{(\alpha,\beta)}(x,y;q)$ holds true:

$$[1-\beta ty]_q^{-m}\exp_q\left[\frac{-\alpha xt}{1-\beta ty}\right]=\sum_{n=0}^{\infty}L_{m,n}^{(\alpha,\beta)}(x,y;q)t^n, \tag{2.2}$$

where $|t|<1,\ |q|<1$.

Proof. Let us denote the left hand sides of (2.2) by W, then by using relation (1.7), we obtain

$$W=\sum_{r=0}^{\infty}\frac{(-1)^r(\alpha xt)^r}{(q,q)_r}[1-\beta ty]_q^{-m-r}, \tag{2.3}$$

appling relation (1.15), we get

$$W=\sum_{r=0}^{\infty}\frac{(-1)^r(\alpha xt)^r}{(q,q)_r}\sum_{n=0}^{\infty}\begin{bmatrix}-m-r\\n\end{bmatrix}_q q^{\binom{n}{2}}(-\beta ty)^n,$$

which by using relation (1.6) becomes

$$W=\sum_{n,r=0}^{\infty}\frac{(-1)^r}{(q,q)_r}\begin{bmatrix}m+r+n-1\\n\end{bmatrix}_q (q^{-m-r})^n(\alpha xt)^r(\beta ty)^n, \tag{2.4}$$

on using relation (1.5), gives

$$W=\sum_{n=0}^{\infty}\frac{(q^m,q)_n(\beta y)^n}{(q,q)_n}\sum_{r=0}^{n}\frac{(-1)^r q^{(-m-r)(n-r)}(q,q)_n}{(q,q)_r(q^m,q)_r(q,q)_{n-r}}\left(\frac{\alpha x}{\beta y}\right)^r t^n$$

$$=\sum_{n=0}^{\infty}\frac{(q^m,q)_n(\beta y)^n}{q^{mn}(q,q)_n}\sum_{r=0}^{n}q^{\binom{r}{2}}\frac{(-1)^r q^{\binom{r}{2}-nr}(q,q)_n}{(q,q)_r(q^m,q)_r(q,q)_{n-r}}\left(q^{m+1}\frac{\alpha x}{\beta y}\right),$$

from relation (1.3), we get

$$W=\sum_{n=0}^{\infty}\frac{\left(q^{m},q\right)_{n}(\beta y)^{n}}{q^{mn}(q,q)_{n}}\sum_{r=0}^{n}q^{\binom{r}{2}}\frac{\left(q^{-n},q\right)_{r}}{(q,q)_{r}\left(q^{m},q\right)_{r}}\left(q^{m+1}\frac{\alpha x}{\beta y}\right)^{r}t^{n}, \quad (2.5)$$

by using definition (2.1), we get the required result (2.2).

Next, we derive some recurrence relations for the polynomials $L_{m,n}^{(\alpha,\beta)}(x,y;q)$ in the form of the following theorems:

Theorem 2.2

The q-analogue Laguerre polynomials of two variables $L_{m,n}^{(\alpha,\beta)}(x,y;q)$ satisfy the following relations:

$$\frac{\partial}{\partial x}L_{m,n}^{(\alpha,\beta)}(x,y;q)=-\alpha\, L_{m+1,n-1}^{(\alpha,\beta)}(x,y;q), \quad (2.6)$$

and

$$\frac{\partial}{\partial y}L_{m,n}^{(\alpha,\beta)}(x,y;q)=m\beta\, L_{m+1,n-1}^{(\alpha,\beta)}(x,y;q)$$

$$-\alpha\beta x\sum_{n=0}^{\infty}(\beta y)^{n-2}\sum_{k=0}^{n-2}\sum_{r=0}^{n-k-2}(-1)^{k}\frac{q^{(-k-1)(n-k-r-2)}\left(q^{k+1};q\right)_{n-k-r-2}\left(q^{m+1};q\right)_{r}}{q^{(m+1)r}(q;q)_{r}(q;q)_{k}(q;q)_{n-k-r-2}}\left(\frac{\alpha x}{\beta y}\right)^{k}. \quad (2.7)$$

Proof.

Differentiating both sides of (2.2) with respect to x, we get

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}L_{m,n}^{(\alpha,\beta)}(x,y;q)t^{n}=-\alpha\, t\sum_{r=0}^{\infty}(-1)^{r}\frac{(\alpha xt)^{r}}{(q;q)_{r}}\left[1-\beta ty\right]_{q}^{-m-r-1},$$

by using relation (1.15), we get

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}L_{m,n}^{(\alpha,\beta)}(x,y;q)t^{n}=-\alpha\, t\sum_{r=0}^{\infty}(-1)^{r}\frac{(\alpha xt)^{r}}{(q;q)_{r}}\sum_{n=0}^{\infty}\begin{bmatrix}-m-r-1\\ n\end{bmatrix}_{q}q^{\binom{n}{2}}(-\beta ty)^{n}, \quad (2.8)$$

from relation (1.6), we obtain

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}L_{m,n}^{(\alpha,\beta)}(x,y;q)t^{n}=-\alpha t\sum_{r=0}^{\infty}(-1)^{r}\frac{(\alpha xt)^{r}}{(q;q)_{r}}\sum_{n=0}^{\infty}\begin{bmatrix}m+r+n\\ n\end{bmatrix}_{q}\left(q^{-m-r-1}\right)^{n}(\beta ty)^{n},$$

which by using relation (1.5), gives

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}L_{m,n}^{(\alpha,\beta)}(x,y;q)t^{n}=-\alpha\sum_{n=0}^{\infty}\sum_{r=0}^{n-1}(-1)^{r}\frac{q^{(-m-r-1)(n-r-1)}(q;q)_{m+n-1}(\alpha x)^{r}(\beta y)^{n-r-1}}{(q;q)_{r}(q;q)_{m+r}(q;q)_{n-r-1}}t^{n}, \quad (2.9)$$

by equating the coefficients of t^{n}, we obtain

$$\frac{\partial}{\partial x}L_{m,n}^{(\alpha,\beta)}(x,y;q)=-\alpha\frac{\left(q^{m+1};q\right)_{n-1}(\beta y)^{n-1}}{q^{(m+1)(n-1)}(q;q)_{n-1}}\sum_{r=0}^{n-1}q^{\binom{r}{2}}(-1)^{r}\frac{q^{\binom{r}{2}-(n-1)r}(q;q)_{n-1}}{(q;q)_{r}\left(q^{m+1};q\right)_{r}(q;q)_{n-r-1}}\left(q^{m+2}\frac{\alpha x}{\beta y}\right)^{r}$$

$$=-\alpha\frac{\left(q^{m+1};q\right)_{n-1}(\beta y)^{n-1}}{q^{(m+1)(n-1)}(q;q)_{n-1}}\sum_{r=0}^{n-1}q^{\binom{r}{2}}\frac{\left(q^{1-n};q\right)_{r}}{(q;q)_{r}\left(q^{m+1};q\right)_{r}}\left(q^{m+2}\frac{\alpha x}{\beta y}\right)^{r},$$

which is the required relation (2.6).

Also, differentiating the both sides of (2.1) with respect to y and using (1.11), we get

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial y}L_{m,n}^{(\alpha,\beta)}(x,y;q)t^{n}=m(\beta t)\left[1-\beta yt\right]_{q}^{-m-1}\exp_{q}\left[-\frac{\alpha xt}{1-\beta yt}\right]$$

$$+\left[1-q\beta yt\right]_q^{-m}\left[\frac{-\alpha\beta\, xt^2}{\left[1-\beta yt\right]_q\left[1-q\beta yt\right]_q}\right]\exp_q\left[-\frac{\alpha xt}{1-\beta yt}\right], \qquad (2.10)$$

by using relation (1.7), we obtain

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial y}L_{m,n}^{(\alpha,\beta)}(x,y;q)t^n = m\beta t\sum_{k=0}^{\infty}(-1)^k\frac{(\alpha xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}m-k-1\\ n\end{bmatrix}_q q^{\binom{n}{2}}(-\beta yt)^n$$

$$-\alpha\beta xt^2\sum_{r=0}^{\infty}\begin{bmatrix}-m-1\\ r\end{bmatrix}_q q^{\binom{r}{2}}(-q\beta y)^r\sum_{k=0}^{\infty}(-1)^k\frac{(\alpha xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}-k-1\\ n\end{bmatrix}_q q^{\binom{n}{2}}(-\beta yt)^n,$$

applying relations (1.6) and (1.5), we find

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial y}L_{m,n}^{(\alpha,\beta)}(x,y;q)t^n = m\beta\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}(-1)^k\frac{q^{(-m-k-1)(n-k-1)}\left(q^{m+1};q\right)_{n-1}(\alpha x)^k(\beta y)^{n-k-1}}{(q;q)_k\left(q^{m+1};q\right)_k(q;q)_{n-k-1}}t^n$$

$$-\alpha\beta x\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}(-1)^k\frac{q^{(-m-1)r+(-k-1)n}\left(q^{k+1};q\right)_n(q;q)_{m+r}(\alpha x)^k(\beta y)^{n+r}}{(q;q)_m(q;q)_r(q;q)_k(q;q)_n}t^{n+k+r+2}, \qquad (2.11)$$

now, by using relation (1.3), we get

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial y}L_{m,n}^{(\alpha,\beta)}(x,y;q)t^n = m\beta\sum_{n=0}^{\infty}\frac{\left(q^{m+1};q\right)_{n-1}(\beta y)^{n-1}}{q^{(m+1)(n-1)}(q;q)_{n-1}}\sum_{k=0}^{n}\frac{\left(q^{1-n};q\right)_k}{(q;q)_k\left(q^{m+1};q\right)_k}\left(\frac{q^{m+2}\alpha x}{\beta y}\right)^k t^n$$

$$-\alpha\beta x\sum_{n=0}^{\infty}(\beta y)^{n-2}\sum_{k=0}^{n-2}\sum_{r=0}^{n-k-2}(-1)^k\frac{q^{(-k-1)(n-k-r-2)}\left(q^{k+1};q\right)_{n-k-r-2}\left(q^{m+1};q\right)_r}{q^{(m+1)r}(q;q)_r(q;q)_k(q;q)_{n-k-r-2}}\left(\frac{\alpha x}{\beta y}\right)^k t^n,$$

by equating the coefficient of t^n, we obtain the required relation (2.7).

Theorem 2.3

The q-analogue Laguerre polynomials of two variables $L_{m,n}^{(\alpha,\beta)}(x,y;q)$ satisfy the following relations:

$$[n+1]_q L_{m,n+1}^{(\alpha,\beta)}(x,y;q)=(m\beta y)L_{m+1,n}^{(\alpha,\beta)}(x,y;q)$$

$$-\alpha x\sum_{n=0}^{\infty}(\beta y)^n\sum_{k=0}^{n}\sum_{r=0}^{n-k}(-1)^k\frac{q^{(-k-1)(n-k-r)}\left(q^{k+1};q\right)_{n-k-r}\left(q^{m+1};q\right)_r}{q^{(m+1)r}(q;q)_r(q;q)_k(q;q)_{n-k-r}}\left(\frac{\alpha x}{\beta y}\right)^k. \qquad (2.12)$$

Proof.

Differentiating the both sides of (2.1) with respect to t and using (1.12), we get

$$\sum_{n=1}^{\infty}[n]_q L_{m,n}^{(\alpha,\beta)}(x,y;q)t^{n-1} = m\beta y\left[1-\beta yt\right]_q^{-m-1}\exp_q\left[-\frac{\alpha xt}{1-\beta yt}\right]$$

$$+\left\{\frac{-\alpha\, x}{\left[1-\beta yt\right]_q\left[1-q\beta yt\right]_q}\right\}\left[1-q\beta yt\right]_q^{-m}\exp_q\left[-\frac{\alpha xt}{1-\beta yt}\right],$$

by using relation (1.7), we get

$$\sum_{n=0}^{\infty}[n+1]_q L_{m,n+1}^{(\alpha,\beta)}(x,y;q)t^n$$

$$= m\beta y\left[1-q\beta yt\right]_q\sum_{k=0}^{\infty}\frac{(-1)^k(\alpha xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}-m-k-1\\ n\end{bmatrix}_q q^{\binom{n}{2}}(-\beta yt)^n$$

$$-\alpha x\sum_{r=0}^{\infty}\begin{bmatrix}-m-1\\ r\end{bmatrix}_q q^{\binom{r}{2}}(-q\beta yt)^r\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{\left(q^{-k-1}\right)^n(q;q)_{k+n}(\alpha x)^k(\beta y)^n}{(q;q)_k(q;q)_k(q;q)_n}t^{n+k}, \quad (2.13)$$

by using relation (1.6), we find

$$\sum_{n=0}^{\infty}[n+1]_q L_{m,n+1}^{(\alpha,\beta)}(x,y;q)t^n = m\beta y\sum_{k=0}^{\infty}\frac{(-1)^k(\alpha x)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}m+k+n\\ n\end{bmatrix}_q\left(q^{-m-k-1}\right)^n(\beta y)^n t^{n+k}$$

$$-\alpha x\sum_{r=0}^{\infty}\begin{bmatrix}m+r\\ r\end{bmatrix}_q\left(q^{-m-1}\right)^r(q\beta y)^r\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{\left(q^{-k-1}\right)^n\left(q^{k+1};q\right)_n(\alpha x)^k(\beta y)^n}{(q;q)_k(q;q)_n}t^{n+k+r},$$

also, by using relations (1.5) and (1.3), we find

$$\sum_{n=0}^{\infty}[n+1]_q L_{m,n+1}^{(\alpha,\beta)}(x,y;q)t^n = m\beta y\sum_{n=0}^{\infty}\frac{\left(q^{m+1};q\right)_n(\beta y)^n}{q^{(m+1)n}(q;q)_n}\sum_{k=0}^{n}q^{\binom{k}{2}}\frac{\left(q^{-n};q\right)_k}{(q;q)_k\left(q^{m+1};q\right)_k}\left(q^{m+2}\frac{\alpha x}{\beta y}\right)^k t^n$$

$$-\alpha x\sum_{n=0}^{\infty}(\beta y)^n\sum_{k=0}^{n}\sum_{r=0}^{n-k}(-1)^k\frac{q^{(-k-1)(n-k-r)}\left(q^{k+1};q\right)_{n-k-r}\left(q^{m+1};q\right)_r}{q^{(m+1)r}(q;q)_r(q;q)_k(q;q)_{n-k-r}}\left(\frac{\alpha x}{\beta y}\right)^k t^n, \quad (2.14)$$

by equating the coefficients of t^n, we get the required relation(2.12).

2. The Generalized q-Analogue Modified Laguerre Polynomials of Two Variables

Now, we introduce generalized q-analogue modified Laguerre polynomial of two variables by means of the following:

$$L_{m,n}^{(\alpha,\beta)}(x,y,a;q)=\frac{\left(q^m;q\right)_n(\beta y)^n}{q^{mn}(q;q)_n}{}_1\phi_1\left(q^{-n},q^m;q^{(a+1)},q^{m+1}\frac{\alpha x}{\beta y}\right). \quad (3.1)$$

We get generating function of the generalized q-analogue Laguerre polynomials in the form of the following theorem:

Theorem 3.1

The following generating function of the generalized q-analogue Laguerre polynomials $L_{m,n}^{(\alpha,\beta)}(x,y,a;q)$ holds true:

$$[1-\beta yt]_q^{-m}E_q\left[\frac{-\alpha xt}{1-\beta yt},a\right]=\sum_{n=0}^{\infty}L_{m,n}^{(\alpha,\beta)}(x,y,a;q)t^n,\quad |t|<1,\ |q|<1. \quad (3.2)$$

Proof. Let us denote the left hand sides of (2.2) by V and using (1.7), we get

$$V=[1-\beta yt]_q^{-m}\sum_{k=0}^{\infty}\frac{(-1)^k q^{a\binom{k}{2}}(\alpha xt)^k}{(q;q)_k}[1-\beta yt]_q^{-k}, \quad (3.3)$$

by using the relation (1.15), we get

$$V=\sum_{k=0}^{\infty}(-1)^k\frac{q^{a\binom{k}{2}}(\alpha xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}-m-k\\ n\end{bmatrix}_q q^{\binom{n}{2}}(-\beta yt)^n,$$

which on using relation (1.6), gives

$$V=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{q^{a\binom{k}{2}+(-m-k)n}(\alpha x)^k}{(q;q)_k}\begin{bmatrix}m+k+n-1\\ n\end{bmatrix}_q(\beta y)^n t^{n+k}, \quad (3.4)$$

on using relation (1.5), we find

$$V=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{q^{a\binom{k}{2}+(-m-k)n}(\alpha x)^k}{(q;q)_k}\frac{(q^m;q)_{k+n}}{(q^m;q)_k(q;q)_n}(\beta y)^n t^{n+k}$$

$$=\sum_{n=0}^{\infty}\frac{(q^m;q)_n(\beta y)^n}{q^{mn}(q;q)_n}\sum_{k=0}^{[n]}(-1)^k\frac{q^{(a+1)\binom{k}{2}+\binom{k}{2}-nk}}{(q;q)_k}\frac{(q;q)_n}{(q^m;q)_k(q;q)_{n-k}}\left(q^{m+1}\frac{\alpha x}{\beta y}\right)^k t^n,$$

by using relation (1.3), we obtain

$$V=\sum_{n=0}^{\infty}\frac{(q^m;q)_n(\beta y)^n}{q^{mn}(q;q)_n}\sum_{k=0}^{\infty}\frac{q^{(a+1)\binom{k}{2}}}{(q;q)_k}\frac{(q^{-n};q)_k}{(q^m;q)_k}\left(q^{m+1}\frac{\alpha x}{\beta y}\right)^k t^n, \tag{3.5}$$

by using definition (3.1), we get the required relation (3.2).

Next, we derive some recurrence relations for the polynomials $L_{m,n}^{(\alpha,\beta)}(x,y,a;q)$ in the form of the following theorems:

Theorem 3.2

The generalized q-analogue Laguerre polynomials of two variables $L_{m,n}^{(\alpha,\beta)}(x,y,a;q)$ satisfy the following relations:

$$\frac{\partial}{\partial x}L_{m,n}^{(\alpha,\beta)}(x,y,a;q)=-\frac{\alpha}{(1-q)}L_{m+1,n-1}^{(\alpha,\beta)}(q^a x,y,a;q), \tag{3.6}$$

and

$$\frac{\partial}{\partial y}L_{m,n}^{(\alpha,\beta)}(x,y,a;q)=m\,L_{m+1,n-1}^{(\alpha,\beta)}(x,y,a;q)$$

$$-\frac{\alpha x t^2}{(1-q)}(\beta y)^{n-2}\sum_{k=0}^{n-2}\sum_{r=0}^{n-k-2}(-1)^k\frac{q^{ak+a\binom{k}{2}}(q^{m+1};q)_r(q^{k+1};q)_{n-k-r-2}}{q^{(m+1)r+(k+1)(n-k-r-2)}(q;q)_r(q;q)_k(q;q)_{n-k-r-2}}\left(\frac{\alpha x}{\beta y}\right)^k. \tag{3.7}$$

Proof.

Differentiating the both sides of (3.2) with respect to x, we get

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}L_{m,n}^{(\alpha,\beta)}(x,y,a;q)t^n=-\frac{\alpha t}{(1-q)}\sum_{k=0}^{\infty}(-1)^k\frac{q^{ak+a\binom{k}{2}}(\alpha xt)^k}{(q;q)_k}(1-\beta yt)^{-1-m-k}, \tag{3.8}$$

by using relation (1.15), we get

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}L_{m,n}^{(\alpha,\beta)}(x,y,a;q)t^n=-\frac{\alpha t}{(1-q)}\sum_{k=0}^{\infty}(-1)^k\frac{q^{ak+a\binom{k}{2}}(\alpha xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}-1-m-k\\ n\end{bmatrix}_q q^{\binom{n}{2}}(-\beta yt)^n,$$

from relation (1.6), we find

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}L_{m,n}^{(\alpha,\beta)}(x,y,a;q)t^n=-\frac{\alpha t}{(1-q)}\sum_{k=0}^{\infty}(-1)^k\frac{q^{ak+a\binom{k}{2}}(\alpha xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}m+k+n\\ n\end{bmatrix}_q q^{(-1-m-k)n}(\beta yt)^n,$$

which by using relation (1.5), we find

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}L_{m,n}^{(\alpha,\beta)}(x,y,a;q)t^n=-\frac{\alpha}{(1-q)}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{q^{ak+a\binom{k}{2}+(-1-m-k)n}(q;q)_{m+k+n}(\alpha x)^k(\beta y)^n}{(q;q)_k(q;q)_{m+k}(q;q)_n}t^{n+k+1}$$

$$=-\frac{\alpha}{(1-q)}\sum_{n=0}^{\infty}\frac{\left(q^{m+1};q\right)_{n-1}(\beta y)^{n-1}}{q^{(m+1)(n-1)}(q;q)_{n-1}}\sum_{k=0}^{n-1}(-1)^k\frac{q^{ak+(a+1)\binom{k}{2}+\binom{k}{2}-(n-1)k}(q;q)_{n-1}}{(q;q)_k\left(q^{m+1};q\right)_k(q;q)_{n-k-1}}\left(\frac{q^{m+2}\alpha x}{\beta y}\right)^k t^n, \quad (3.9)$$

by equating the coefficient of t^n , we obtain

$$\frac{\partial}{\partial x}L_{m,n}^{(\alpha,\beta)}(x,y,a;q)=-\frac{\alpha}{(1-q)}\frac{\left(q^{m+1};q\right)_{n-1}(\beta y)^{n-1}}{q^{(m+1)(n-1)}(q;q)_{n-1}}\sum_{k=0}^{\infty}\frac{q^{(a+1)\binom{k}{2}}\left(q^{1-n};q\right)_k}{(q;q)_k\left(q^{m+1};q\right)_k}\left(\frac{q^{a+m+2}\alpha x}{\beta y}\right)^k,$$

which the required relation (3.6).

Also, differentiating the both sides of (3.2) with respect to y and using (1.11), we get

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial y}L_{m,n}^{(\alpha,\beta)}(x,y,a;q)t^n=m\,\beta t(1-\beta yt)^{-m-1}E_q\left[-\frac{\alpha xt}{1-\beta yt},a\right]$$
$$+(1-q\beta yt)^{-1-m}\left[\frac{-\alpha xt^2}{(1-q)(1-\beta yt)}\right]E_q\left[-\frac{q^a\alpha xt}{1-\beta yt},a\right], \quad (3.10)$$

by using relation (1.7), we obtain

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial y}L_{m,n}^{(\alpha,\beta)}(x,y,a;q)t^n=m\beta\,t\sum_{k=0}^{\infty}(-1)^k\frac{q^{a\binom{k}{2}}(\alpha xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}-m-k-1\\n\end{bmatrix}_q q^{\binom{n}{2}}(-\beta yt)^n$$
$$-\frac{\alpha xt^2}{(1-q)}\sum_{r=0}^{\infty}\begin{bmatrix}-m-1\\r\end{bmatrix}_q q^{\binom{r}{2}}(-q\beta yt)^r\sum_{k=0}^{\infty}(-1)^k\frac{q^{ak+a\binom{k}{2}}(\alpha xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}-k-1\\n\end{bmatrix}_q q^{\binom{n}{2}}(-\beta yt)^n,$$

by using relation (1.6), we find

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial y}L_{m,n}^{(\alpha,\beta)}(x,y,a;q)t^n=m\,t\sum_{k=0}^{\infty}(-1)^k\frac{q^{a\binom{k}{2}}(\alpha xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}m+k+n\\n\end{bmatrix}_q q^{(-m-k-1)n}(\beta yt)^n$$
$$-\frac{\alpha xt^2}{(1-q)}\sum_{r=0}^{\infty}\begin{bmatrix}m+r\\r\end{bmatrix}_q q^{(-m-1)r}(q\beta yt)^r\sum_{k=0}^{\infty}(-1)^k\frac{q^{ak+a\binom{k}{2}}(\alpha xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}k+n\\n\end{bmatrix}_q q^{(-k-1)n}(\beta yt)^n, \quad (3.11)$$

from relations (1.5) and (1.3), we obtain

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial y}L_{m,n}^{(\alpha,\beta)}(x,y,a;q)t^n=m\beta\sum_{n=0}^{\infty}\frac{\left(q^{m+1};q\right)_{n-1}(\beta y)^{n-1}}{q^{(m+1)(n-1)}(q;q)_{n-1}}\sum_{k=0}^{n-1}(-1)^k\frac{q^{(a+1)\binom{k}{2}}\left(q^{n-1};q\right)_k}{(q;q)_k\left(q^{m+1};q\right)_k}\left(\frac{q^{m+2}\alpha x}{\beta y}\right)^k t^n$$
$$-\frac{\alpha x}{(1-q)}\sum_{n=0}^{\infty}(\beta y)^{n-2}\sum_{k=0}^{n-2}\sum_{r=0}^{n-k-2}(-1)^k\frac{q^{ak+a\binom{k}{2}}\left(q^{m+1};q\right)_r\left(q^{k+1};q\right)_{n-k-r-2}}{q^{(m+1)r+(k+1)(n-k-r-2)}(q;q)_r(q;q)_k(q;q)_{n-k-r-2}}\left(\frac{\alpha x}{\beta y}\right)^k t^n,$$

by equating the coefficient of t^n , we obtain the required relation (3.7).

Theorem 3.3

The generalized q-analogue Laguerre polynomials of two variable $L_{m,n}^{(\alpha,\beta)}(x,y,a;q)$ satisfy the following relations:

$$[n+1]_q L_{m,n+1}^{(\alpha,\beta)}(x,y,a;q)=m\beta yL_{m+1,n}^{(\alpha,\beta)}(x,y,a;q)$$

$$-\frac{\alpha x(\beta y)^n}{(1-q)}\sum_{k=0}^{n}\sum_{r=0}^{n-k}(-1)^k\frac{q^{ak+a\binom{k}{2}}\left(q^{m+1};q\right)_r\left(q^{k+1};q\right)_{n-k-r}}{q^{(m+1)r+(k+1)(n-k-r)}(q;q)_r(q;q)_k(q;q)_{n-k-r}}\left(\frac{\alpha x}{\beta y}\right)^k. \quad (3.12)$$

Proof.

Differentiating the both sides of (3.1) with respect to t, we get

$$\sum_{n=0}^{\infty}[n]L_{m,n}^{(\alpha,\beta)}(x,y,a;q)t^{n-1}=m\beta y(1-\beta yt)^{-m-1}E_q\left[-\frac{\alpha xt}{1-\beta yt},a\right]$$
$$+\left\{\frac{-\alpha x}{(1-q)(1-\beta yt)}\right\}(1-q\beta yt)^{-m-1}E_q\left[-\frac{q^a\alpha xt}{1-\beta yt},a\right], \quad (3.13)$$

by using relation (1.7), we get

$$\sum_{n=0}^{\infty}[n+1]_q L_{m,n+1}^{(\alpha,\beta)}(x,y,a;q)t^n=m\beta y\sum_{k=0}^{\infty}(-1)^k\frac{q^{a\binom{k}{2}}(\alpha xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}-m-k-1\\ n\end{bmatrix}_q q^{\binom{n}{2}}(-\beta yt)^n$$

$$-\frac{\alpha x}{(1-q)}\sum_{r=0}^{\infty}\begin{bmatrix}-m-1\\ r\end{bmatrix}_q q^{\binom{r}{2}}(-\beta yt)^r\sum_{k=0}^{\infty}(-1)^k\frac{q^{ak+a\binom{k}{2}}(\alpha xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}-k-1\\ n\end{bmatrix}_q q^{\binom{n}{2}}(-\beta yt)^n,$$

which by using relation (1.6), we find

$$\sum_{n=0}^{\infty}[n+1]_q L_{m,n+1}^{(\alpha,\beta)}(x,y,a;q)t^n=m\beta y\sum_{k=0}^{\infty}(-1)^k\frac{q^{a\binom{k}{2}}(\alpha x)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}m+k+n\\ n\end{bmatrix}_q q^{(-m-k-1)n}(\beta y)^n t^{n+k}$$

$$-\frac{\alpha x}{(1-q)}\sum_{r=0}^{\infty}\begin{bmatrix}m+r\\ r\end{bmatrix}_q q^{(-m-1)r}(\beta yt)^n\sum_{k=0}^{\infty}(-1)^k\frac{q^{ak+a\binom{k}{2}}(\alpha x)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}k+n\\ n\end{bmatrix}_q q^{(-k-1)n}(\beta y)^n t^{n+k},$$

by using relation (1.5), we find

$$\sum_{n=0}^{\infty}[n+1]_q L_{m,n+1}^{(\alpha,\beta)}(x,y,a;q)t^n$$

$$=m\beta y\sum_{n=0}^{\infty}\frac{\left(q^{m+1};q\right)_n(\beta y)^n}{q^{(m+1)n}(q;q)_n}\sum_{k=0}^{n}(-1)^k\frac{q^{(a+1)\binom{k}{2}+\binom{k}{2}-nk}(q;q)_n}{(q;q)_k\left(q^{m+1};q\right)_k(q;q)_{n-k}}\left(\frac{q^{m+2}\alpha x}{\beta y}\right)^k t^n$$

$$-\frac{\alpha x}{(1-q)}\sum_{n=0}^{\infty}(\beta y)^n\sum_{k=0}^{n}\sum_{r=0}^{n-k}(-1)^k\frac{q^{ak+a\binom{k}{2}}\left(q^{m+1};q\right)_r\left(q^{k+1};q\right)_{n-k-r}}{q^{(m+1)r+(k+1)(n-k-r)}(q;q)_r(q;q)_k(q;q)_{n-k-r}}\left(\frac{\alpha x}{\beta y}\right)^k t^n,$$

from relation (1.3), we obtain

$$\sum_{n=0}^{\infty}[n+1]_q L_{m,n+1}^{(\alpha,\beta)}(x,y,a;q)t^n=m\beta y\sum_{n=0}^{\infty}\frac{\left(q^{m+1};q\right)_n(\beta y)^n}{q^{(m+1)n}(q;q)_n}\sum_{k=0}^{n}\frac{q^{(a+1)\binom{k}{2}}\left(q^{-n};q\right)_n}{(q;q)_k\left(q^{m+1};q\right)_k}\left(\frac{q^{m+2}\alpha x}{\beta y}\right)^k t^n$$

$$-\frac{\alpha x}{(1-q)}\sum_{n=0}^{\infty}(\beta y)^n\sum_{k=0}^{n}\sum_{r=0}^{n-k}(-1)^k\frac{q^{ak+a\binom{k}{2}}\left(q^{m+1};q\right)_r\left(q^{k+1};q\right)_{n-k-r}}{q^{(m+1)r+(k+1)(n-k-r)}(q;q)_r(q;q)_k(q;q)_{n-k-r}}\left(\frac{\alpha x}{\beta y}\right)^k t^n,$$

by equating the coefficients of t^n , we get the required relation(3.12).

results: We introduced the q-analogue modified Laguerre and generalized modified Laguerre polynomials of two variables . Some recurrence relations for these polynomials are derived; (see (2.1), (2.6), (2.7), (2.12), (3.1), (3.6), (3.7) and (3.12)).

Conclusion:

The generalized form of q-analogue modified Laguerre and generalized modified Laguerre polynomials of two variables are introduced and some of its properties are established in this paper. q- analogue modified Laguerre and generalized modified Laguerre polynomials are important, whose applications and generalizations arise in many applications such as quantumgroup (oscillator algebra, etc.), *q*-harmonic oscillator and coding theory.

References:

1- Dottoli, G. ,(2000), Generalized polynomials, operational identities and their applications, Journal of Computational and Applied Mathematics, 118, 111-123.

2- Exton, H. (1983), q-Hypergeometric Functions and Applications. Ellis Horwoo, Chichester.

3- Gasper,G. and Rahman, M. ,(2004), Basic hypergemetric series, second ed. Cambridge university Press.

4- Khan, S. and Yasmin, G. ,(2002), On generating relations for the two variable modified Laguerre polynomials, Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie, Nouvelle Série, Vol. 45 (93), No. 1/2, pp. 29-37.

5- Koekoek, R. and Swarttouw, R. F. ,(1998), The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue. Report no. 98-17, TU-Delft.

6- Purohit, S.D. and Raina, R.K. ,(2010), Generalized q-Taylor's series and applications, General Mathematics Vol.18, No. 3, 19-28.

7- Rainville, E. D.,(1960), Special Functions, The Macmillan, New York, NY, USA.

8- Rajkovic', P. and Marinkovic', S., (2001), On Q-analogies of generalized Hermite'spolynomials , Presented at the IMC. 277-283.

كثيرات حدود معدلة للاجير و لاجير المعدلة المعممة ذات متغيرين من النوع كيو

فضل بن فضل محسن فضل صالح ناصر السرحي

الملخص

كثيرات حدود لاجير من النوع كيو هي مهمة في كثيرات حدود المتعامدة من النوع كيو حيث تظهر تطبيقاتها وتعميمها في العديد من التطبيقات مثل نظرية الكم (الجبر المتذبذب) والمذبذب المتناسق من النوع كيو ونظرية التشفير. في هذه الورقة قدمنا كثيرات حدود معدلة للاجير و لاجير المعممة ذات متغيرين من النوع كيو وأيضاً أثبتنا العلاقات التكرارية لهما.

الكلمات المفتاحية: كثيرات حدود لاجير المعممة المعدلة ذات متغيرين من النوع كيو، العلاقات التكرارية.