A Chebyshev Spectral Collocation Method for Solving Kdv-Burgers Equation

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Abstract

A mixed spectral/ Runge-Kutta method is used to obtain numerical solutions of Kortewege–de Vries–Burgers' (KdVB) equation. The suggested method based on Chebyshev spectral collocation is used with Runge-Kutta method of order four. This technique is accomplished by starting with a Chebyshev approximation for the higher order derivatives in the x-direction and generating approximations to the lower derivatives through successive integrations of the highest-order derivative. The proposed technique reduces the problem to a system of ordinary differential equations in the t-direction. The Runge-Kutta method of order four is used to solve this system. Excellent numerical results have been obtained compared with the exact solution.

Keywords: Numerical solutions; Chebyshev spectral collocation method; Kortewege–de Vries–B urgers' (KdVB) equation; Runge-Kutta method.

Introduction:

In this paper, we will consider the Kortewege–de Vries–Burgers' (KdVB) equation**:**

$$u_t + \varepsilon u u_x - \nu u_{xx} + \mu u_{xxx} = 0 \qquad (1)$$

where ε, ν and μ are positive parameters. When the parameter $\nu = 0$, Eq.(1) will be the KdV equation and when the parameter $\mu = 0$, Eq.(1) will be the Burgers equation.

Eq.(1) was derived by Su and Gardner [15]. This equation arises in various physical phenomena, see for example [5, 8 and 9].

Many authors have solved KdVB equation in recent years. Arshed et al. [1], a meshless technique was used to solve KdVB equation. Bhatta [2] used modified Bernstein polynomial to solve KdVB equation. Darvishi et al. [3] implemented spectral collocation method to solve KdVB equation. Haq et al. [6] solved the KdVB equation using a mesh-free method. Jafari and Gharbavy [7] solved KdVB equation using optimal homotopy asymptotic method. In Kaya [10], Adomian decomposition method was used to solve KdVB equation. Khater et al. [11] presented a spectral collocation method based on differentiated Chebyshev polynomials to obtain numerical solutions for the KdVB. Sajjadian [12], sinc-collocation method applied to solve KdVB equation. Shi et al. [13], a high-order compact – CIP scheme was used to solve KdVB equation. Soliman [14] presented Variational iteration method to solve KdVB equation. Talaat and El-Danaf [16]. Zaki [17] used quantic B-spline finite elements scheme to solve KdVB equation.

In this paper we used spectral/ Runge-Kutta method to get the numerical solutions of Kortewege–de Vries–Burgers' (KdVB) equation. This paper is organized as follows: In the following section the model problem is described. The description method of mixed method is presented in section 3. Numerical results are presented in section 4. Finally, in section 5, the conclusion is presented.

A model problem:

Consider the Kortewege–de Vries–Burgers' (KdVB) equation [14]

$$u_t + \varepsilon u u_x - \nu u_{xx} + \mu u_{xxx} = 0, \qquad (x,t) \in D \times [0,T] \qquad (2)$$

With initial condition

$$u(x,0) = f(x), \; x \in D \qquad (3)$$

and boundary conditions

$$\begin{aligned} &u(x,t) = g(t), \\ &u_x(x,t) = h(t), \quad (x,t) \in \delta D \times [0,T], \end{aligned} \qquad (4)$$

where $D = \{x : 0 \le x \le 100\}$ and δD is its boundary.

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Description of the method:

To solve the Kortewege–de Vries–Burgers' (KdVB) equation (2) with the initial and boundary conditions (3) and (4), in the domain [0,100]×[0,T]. Firstly, using the algebraic mapping $y=\frac{1}{50}x-1,$ Which maps the boundary region onto the finite domain $[-1,1]$, and equation (2) with the conditions (3) and (4) is transformed to

$$u_t+\frac{1}{50}\varepsilon u u_y-\frac{1}{50^2}\nu u_{yy}+\frac{1}{50^3}\mu u_{yyy}=0,\qquad (y,t)\in D^*\times[0,T] \tag{5}$$

$$u(y,-1)=f(y),\quad y\in D^* \tag{6}$$

$$u(y,t)=g(t),$$
$$u_y(y,t)=h(t),\quad (y,t)\in\delta D^*\times[0,T], \tag{7}$$

where $D^*=\{y:-1\le y\le 1\}$ and δD^* is its boundary.

Now, setting $u_{yyy}(y,t)=\phi(y,t)$. Then

$$u_{yy}(y,t)=\int_{-1}^{y}\phi(y,t)\,dy+c_1, \tag{8}$$

$$u_y(y,t)=\int_{-1}^{y}\int_{-1}^{y}\phi(y,t)\,dy\,dy+(y+1)c_1+c_2 \tag{9}$$

$$u(y,t)=\int_{-1}^{y}\int_{-1}^{y}\int_{-1}^{y}\phi(y,t)\,dy\,dy\,dy+\frac{1}{2}(y+1)^2c_1+(y+1)c_2+c_3 \tag{10}$$

where c_1, c_2 and c_3 are computed from conditions (7).

Now, we give approximations for Eqs. (8)-(10) are as follows:

$$u_{yy}(y_i,t)=\sum_{j=0}^{N}\ell_{ij}^{(2)}\phi(y_j,t)+d_i^{(2)}(t), \tag{11}$$

$$u_y(y_i,t)=\sum_{j=0}^{N}\ell_{ij}^{(1)}\phi(y_j,t)+d_i^{(1)}(t) \tag{12}$$

$$u(y_i,t)=\sum_{j=0}^{N}\ell_{ij}\phi(y_j,t)+d_i(t) \tag{13}$$

for all $i,j=0,1,\cdots,N$ and $y_i=-\cos(i\pi/N),\ i=0,1,\cdots,N$

$$\ell_{ij}=b_{ij}^{(3)}-\frac{1}{4}(y_i+1)^2\,b_{Nj}^{(3)},$$

$$\ell_{ij}^{(1)}=b_{ij}^{(2)}-\frac{1}{2}(y_i+1)\,b_{Nj}^{(3)},$$

$$\ell_{ij}^{(2)}=b_{ij}-\frac{1}{2}b_{Nj}^{(3)},$$

$$d_i=\frac{1}{4}(y_i+1)^2(g_1(t)-g_0(t)-2h_0(t))+(y_i+1)h_0(t)+g_0(t),$$

$$d_i^{(1)}=\frac{1}{2}(y_i+1)(g_1(t)-g_0(t)-2h_0(t))+h_0(t),$$

$$d_i^{(2)}=\frac{1}{2}(g_1(t)-g_0(t)-2h_0(t))$$

and

$$b_{ij}^{(n)} = \frac{(y_i - y_j)^{n-1}}{(n-1)!} b_{ij}, \quad i, j = 0,1,...,N, \quad n = 2,3 \ ,$$

where g_0, h_0 and g_1 are known functions given in the boundary conditions (4) at the endpoints of the boundary respectively, and b_{ij} are the elements of the matrix B, as defined by El-Gendi [4].

By using Eqs. (11) - (13), Eq. (5) is transformed to the following system of ordinary differential equations in the t -direction:

$$\sum_{j=1}^{N-1} \ell_{ij} \frac{d}{dt}\phi(y_j,t) + \frac{d}{dt} d_i(t) + \frac{1}{50}\varepsilon\Big(\sum_{j=1}^{N-1}\ell_{ij}\phi(y_j,t) + d_i(t)\Big)\Big(\sum_{j=1}^{N-1}\ell_{ij}^{(1)}\phi(y_j,t) + d_i^{(1)}(t)\Big) - \frac{1}{50^2}\nu\Big(\sum_{j=1}^{N-1}\ell_{ij}^{(2)}\phi(y_j,t) + d_i^{(2)}(t)\Big) + \frac{1}{50^3}\mu\phi(y_i,t) = 0 \quad (14)$$

This system can be solved by Runge-Kutta method of order four. We write this system as

$$\sum_{j=1}^{N-1}\ell_{ij}\phi_j'(t) = F_i(t,\phi(t)) \ ,$$

$$\phi_j(0) = \phi_{j,0} \ , \quad (15)$$

where $t \in [0,b], \ i = 1,\dots,N-1$.

Then system (15) can be rewritten in the following form

$$\sum_{j=1}^{N-1}\ell_{ij}\phi'(t) = F(t,\phi(t)) \ ,$$

$$\phi(0) = \phi_0 \quad (16)$$

where

$$\phi(t) = [\phi_1(t),\phi_2(t),\dots,\phi_{N-1}(t)]^T, \ \phi'(t) = [\phi_1'(t),\phi_2'(t),\dots,\phi_{N-1}'(t)]^T,$$

$$\phi(0) = [\phi_1(0),\phi_2(0),\dots,\phi_{N-1}(0)]^T, \ F(t,\phi(t)) = [F_1(t,\phi(t)),F_2(t,\phi(t)),\dots,F_{N-1}(t,\phi(t))]^T$$

and

$$F_i(t,\phi(t)) = \frac{1}{50^2}\nu\Big(\sum_{j=1}^{N-1}\ell_{ij}^{(2)}\phi(y_j,t) + d_i^{(2)}(t)\Big) - \frac{1}{50}\varepsilon\Big(\sum_{j=1}^{N-1}\ell_{ij}\phi(y_j,t) + d_i(t)\Big)\Big(\sum_{j=1}^{N-1}\ell_{ij}^{(1)}\phi(y_j,t) + d_i^{(1)}(t)\Big) - \frac{1}{50^3}\mu\phi(y_i,t) - \frac{d}{dt}d_i(t) \ ,$$

which is a system of ordinary differential equations in time t (16). We use the Runge-Kutta method of order four to solve this system. The Runge-Kutta method of order four is given by:

$$\sum_{j=1}^{N-1}\ell_{ij}\phi(y_j,t_{m+1}) = \sum_{j=1}^{N-1}\ell_{ij}\phi(y_j,t_m) + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \ , \quad (17)$$

where

$$k_1 = F_i(t_m,\phi(t_m)),$$

$$k_2 = F_i(t_m + \frac{h}{2},\phi(t_m) + \frac{1}{2}k_1),$$

$$k_3 = F_i(t_m + \frac{h}{2},\phi(t_m) + \frac{1}{2}k_2),$$

$$k_4 = F_i(t_m + h,\phi(t_m) + k_3),$$

and $t_m = mh, \ h = \dfrac{b}{M}, \quad m = 0,1,\dots,M.$

4. Numerical results:

In this section, we solve the Kortewege–de Vries–Burgers' (KdVB) equation (2), which has the following exact solution [8]:

$$u(x,t) = -\frac{6\nu^2}{25\mu}\left[1+\tanh\left(\frac{\nu}{10\mu}\left(x+\frac{6\nu^2}{25\mu}t\right)\right)-\frac{1}{2}\sec h^2\left(\frac{\nu}{10\mu}\left(x+\frac{6\nu^2}{25\mu}t\right)\right)\right] \quad (18)$$

The initial and boundary conditions (3) and (4) were taken from exact solution (18).

We compare our numerical results with the exact solution for $\varepsilon = 1$ and different values of time t, and parameters ν, μ, (See Tables 1-3).

Table1, shows absolute errors in the results obtained by the proposed method for various values of x, t and $\nu = 0.001, \mu = 1, h = 0.001$ with N=6 and M= 10.

In Table2, we present absolute errors for ν =0.001, h=0.0001 and various values of μ with N=12 and M= 20.

The maximum absolute errors for h=0.001 and various values of ν and μ with N=12 and M= 20 are shown in Table3.

Table 1: Comparison of the exact and numerical solutions for ν = 0.001, μ=1and h=0.001

t	*x*	Exact sol.	Numerical sol.	Absol. Error
0.0	25.0	-1.206008E-07	-1.199792E-07	6.215336E-10
	50.0	-1.212030E-07	-1.199883E-07	1.214750E-09
	75.0	-1.218067E-07	-1.200007E-07	1.805992E-09
0.2	25.0	-1.206008E-07	-1.199792E-07	6.215358E-10
	50.0	-1.212030E-07	-1.199887E-07	1.214270E-09
	75.0	-1.218067E-07	-1.200022E-07	1.804545E-09
0.4	25.0	-1.206008E-07	-1.199792E-07	6.215379E-10
	50.0	-1.212030E-07	-1.199892E-07	1.213791E-09
	75.0	-1.218067E-07	-1.200036E-07	1.803097E-09
0.6	25.0	-1.206008E-07	-1.199792E-07	6.215401E-10
	50.0	-1.212030E-07	-1.199897E-07	1.213311E-09
	75.0	-1.218067E-07	-1.200051E-07	1.801649E-09
0.8	25.0	-1.206008E-07	-1.199792E-07	6.215423E-10
	50.0	-1.212030E-07	-1.199902E-07	1.212832E-09
	75.0	-1.218067E-07	-1.200065E-07	1.800201E-09
1.0	25.0	-1.206008E-07	-1.199792E-07	6.215444E-10
	50.0	-1.212030E-07	-1.199906E-07	1.212352E-09
	75.0	-1.218067E-07	-1.200080E-07	1.798753E-09

Table 2: Absolute errors for ν =0.001, h=0.0001 and various values of μ

		t				
x	μ	0.05	0.1	0.3	0.7	0.9
25.0	1	6.179691E-10	6.180942E-10	6.185949E-10	6.195968E-10	6.200979E-10
50.0		1.208404E-09	1.208005E-09	1.206410E-09	1.203219E-09	1.201623E-09
75.0		1.798267E-09	1.797149E-09	1.792673E-09	1.783720E-09	1.779242E-09
25.0	2	1.543988E-10	1.544301E-10	1.545553E-10	1.548060E-10	1.549943E-10
50.0		3.017279E-10	3.016282E-10	3.012294E-10	3.004312E-10	2.998323E-10
75.0		4.487294E-10	4.484497E-10	4.473306E-10	4.450909E-10	4.434098E-10
25.0	5	2.469483E-11	2.469984E-11	2.471990E-11	2.476011E-11	2.478026E-11
50.0		4.824055E-11	4.822460E-11	4.816074E-11	4.803288E-11	4.796887E-11
75.0		7.171599E-11	7.167122E-11	7.149202E-11	7.113305E-11	7.095327E-11

Table 3: Maximum absolute error for h=0.001 and various values of ν and μ

		t				
ν	μ	0.0	0.2	0.4	0.8	1.0
0.001	0.1	2.432458E-07	2.378471E-07	2.324469E-07	2.216420E-07	2.162373E-07
	1.0	2.336337E-09	2.282212E-09	2.227932E-09	2.118906E-09	2.064159E-09
	5.0	9.308538E-11	9.090467E-11	8.869263E-11	8.417322E-11	8.186518E-11
0.0001	0.1	2.336336E-10	2.282298E-10	2.228244E-10	2.120090E-10	2.065990E-10
	1.0	2.325976E-12	2.271846E-12	2.217560E-12	2.108521E-12	2.053767E-12
	5.0	9.300199E-14	9.082123E-14	8.860912E-14	8.408951E-14	8.178135E-14
0.00001	0.1	2.325976E-13	2.271933E-13	2.217874E-13	2.109710E-13	2.055605E-13
	1.0	2.324934E-15	2.270803E-15	2.216516E-15	2.107476E-15	2.052721E-15
	5.0	9.299364E-17	9.081288E-17	8.860076E-17	8.408113E-17	8.177295E-17

Conclusion:

The mixed spectral/ Runge-Kutta method gives excellent numerical solutions of Kortewege–de Vries–Burgers' (KdVB) equation compared with the exact solutions as shown in Tables (1)-(3). Computations of the paper have been carried out using the FORTRAN.

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طريقة تجميع تشبيشف الطيفي لحل معادلة كورتوج دي-فرايس- بيرجر

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الملخص

في هذا البحث قدمنا خليطاً من طريقة الطيف / طريقة رونج- كوتا للحصول على الحلول العددية لمعادلة كورتوج دي–فرايس- بيرجر. هذه الطريقة تعتمد على طريقة تجميع تشبيشف الطيفي مع طريقة رونج- كوتا من الرتبة الرابعة. هذا التكنيك يبدأ بإعطاء تقريب تشبيشف لأعلى مشتقة جزئية في اتجاه x والحصول على تقريبات للمشتقات التفاضلية الأقل وذلك من خلال التكامل المتتالي للمشتقات الأعلى. التكنيك المقترح يحول المسألة إلى نظام من المعادلات التفاضلية العادية في اتجاه t. باستخدام طريقة رونج–كوتا من الرتبة الرابعة نحل هذا النظام. نتائج عددية ممتازة حصلنا عليها مقارنتها مع الحل المضبوط.

الكلمات المفتاحية: الحلول العددية، طريقة تجميع تشبيشف الطيفي، معادلة كورتوج دي–فرايس – بيرجر ، طريقة رونج–كوتا.